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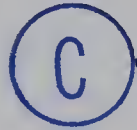
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PROBLEMS IN COMBINATORIAL AND
ADDITIVE NUMBER THEORY

by



JAMES RIDDELL

A THESIS

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The undersigned certify that they have read,
and recommend to the Faculty of Graduate Studies for
acceptance, a thesis entitled "PROBLEMS IN COMBINATORIAL
AND ADDITIVE NUMBER THEORY", submitted by James Riddell in
partial fulfilment of the requirements for the degree of
Doctor of Philosophy.

ABSTRACT

This thesis deals with problems on sets of numbers containing not more than a given number of terms in progression (arithmetic in Chapter I, geometric in Chapter II), and with a problem on bases for sets of integers.

In Chapter I we find a lower estimate for $g_\ell(n)$, the largest number g with the property that one can always select g elements from any set of n reals so that no ℓ of the numbers chosen are in arithmetic progression. We prove that

$$g_\ell(n) > 4^{1/\ell} n^{1-2/\ell} - 1.$$

We also consider the problem of finding how many elements can be chosen from an arbitrary set of n integers so that no three of the chosen integers are in arithmetic progression. We find that if the n integers lie in $[1, n^a]$, where $a \geq 1$, then we can choose at least

$$n^{1 - 3\sqrt{2a \log 2} / \sqrt{\log n}} - 3 \log 2 / \log n$$

elements. We also show that in any interval of length at least n^3 , almost all sets of n integers from that interval contain a progression-free subset of cardinality at least

$$n^{1 - (3\sqrt{6 \log 2} + \varepsilon) / \sqrt{\log n}}.$$

Finally in Chapter I we consider a related geometrical problem.

In Chapter II we obtain an improvement over a result of

(ii)

R. A. Rankin on the density of sets of positive integers that contain no n terms in geometric progression ($n \geq 4$). Our results take different forms according as n is prime, odd and composite, or even. For n prime we construct a set whose density is

$$\zeta(n) / [\zeta(n-1) \zeta\{(n-1)n\}].$$

To obtain the densities in the other cases we prove the following result (Lemma 2.1): If E is any set of non-negative integers containing 0, and $Q(E)$ is the set of all integers N of the form $N = \prod_{i=1}^{\infty} p_i^{a_i}$, where p_i is the i -th prime and each a_i is chosen from E , then $\lim_{x \rightarrow \infty} [Q(E)](x)/x$ exists; that is, $Q(E)$ has a density. We also find for each $n \geq 3$ an upper estimate for the possible density of sets containing no n -term geometric progression.

In Chapter III we find a new lower estimate for $k_h(n)$, the smallest number of reals that can be used to form an h -basis for n . A set $A: a_1 = 0 < a_2 < a_3 < \dots < a_k \leq n$ is called an h -basis for n if each of $0, 1, 2, \dots, n$ can be expressed as the sum of h summands a_i from A , with repetition of summands allowed. While earlier results were obtained using separate arguments for $h = 2$ and for $h > 2$, we handle all cases $h \geq 2$ with a common argument.

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I am further grateful to Professor Moser for writing many letters recommending me for scholarships and other financial aid. I would like also to thank Professor E. S. Keeping, Dr. H. F.J. Lowig, and Dr. J. R. Trollope for their similar undertakings. These recommendations brought me financial help from the following institutions, to which I owe my thanks: the University of Alberta, the Province of Alberta, and the Government of Canada through the Canada Student Loans Plan.

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CHAPTER I

ON SETS OF NUMBERS CONTAINING NO ℓ TERMS IN ARITHMETIC PROGRESSION

§1.1 Introduction

If S is a set of n real numbers, then among the subsets of S that contain no ℓ -term arithmetic progression there is one of maximum cardinality $G_\ell(S)$. Let $g_\ell(n) = \min_S G_\ell(S)$, the minimum being taken over all sets S of n reals. In §1.2 we find a lower estimate for $g_\ell(n)$, and also consider cases in which S is restricted to be a set of integers. In §1.3 we consider a related geometrical problem.

Considerable study has been made of the case $S = \{1, 2, \dots, n\}$. Let $r_\ell(n) = G_\ell(\{1, 2, \dots, n\})$. Szekeres conjectured that $r_\ell(n) = o(n)$, and this has been proved in the case $\ell = 3$ by Roth [18], who showed that $r_3(n) < cn/\log\log n$. Szekeres had conjectured also that for given ℓ there is a number a in $(0, 1)$ such that $r_\ell(n) = O(n^a)$, but this was proved false by Salem and Spencer [19], who proved that for $\varepsilon > 0$,

$$r_3(n) > n^{1 - (\log 2 + \varepsilon)/\log\log n}$$

provided n is sufficiently large. Behrend [3], Moser [11], and Rankin [14] improved this estimate, the latter showing that if $\ell > 2^k$ (k a positive integer), $\varepsilon > 0$, and $c = (k+1) \cdot 2^{k/2} (\log 2)^{k/(k+1)} (1+\varepsilon)$, then

$$(1.1) \quad r_\ell(n) > n^{1 - c/(\log n)^{k/(k+1)}}$$

provided $n > N(\varepsilon, k)$.

The study of the function $r_\ell(n)$ began with Erdős and Turán [7], who were interested in obtaining an upper estimate for a function of van der Waerden. The latter proved [21] that given positive integers k and ℓ , there exists an integer W such that if $1, 2, \dots, W$ are partitioned into k or fewer classes, at least one class contains an ℓ -term arithmetic progression. Let $W(k, \ell)$ be the smallest such integer W . If one could find an integer n such that $r_\ell(n) \leq n/k$, then it would follow that $W(k, \ell) \leq n$. For an account of these problems see Erdős [6, pp. 220-223], where the above conjectures of Szekeres are mentioned.

§1.2 On $g_\ell(n)$ and progression-free sets of integers

In this section we prove the three theorems stated below. By a progression-free set will be meant a set containing no 3-term arithmetic progression.

Theorem 1.1 Let $2 < \ell \leq n$. Then

$$(1.2) \quad g_\ell(n) > 4^{1/\ell} n^{1-2/\ell} - 1.$$

Since $g_\ell(n) \leq r_\ell(n)$, (1.2) provides an estimate for $r_\ell(n)$, and if ℓ is not too small compared with n , a larger estimate than (1.1). In fact, if $\ell > 2 \log n$, then (1.2) implies

$$g_\ell(n) > n^{1-2/\ell} > n/e = n^{1-1/\log n} > n^{1-1/(\log n)^{k/(k+1)}}$$

for any positive integer k and this exceeds the estimate (1.1).

Theorem 1.2 Let $a \geq 1$ be given. Then any set of n integers in $[1, n^a]$ contains a progression-free subset whose cardinality is at least

$$(1.3) \quad n^{1 - \sqrt{3} \cdot 2a \log 2 / \sqrt{\log n}} - 3 \log 2 / \log n.$$

To prove Theorem 1.2 we adapt an argument of Moser [10]. The exponent in (1.3) will be nearly 1 if $a \leq (\log n)/f(n)$ where $f(n) \rightarrow \infty$ as $n \rightarrow \infty$. If $a \geq c \log n$ the estimate (1.3) is of no value, but we assert

Theorem 1.3 For any interval at least as large as $[1, n^3]$ and any $\varepsilon > 0$, almost all sets of n integers in that interval contain a progression-free subset whose cardinality is at least

$$n^{1 - (3 \sqrt{6} \log 2 + \varepsilon) / \sqrt{\log n}}$$

provided n is sufficiently large.

(By almost all sets we mean all but $o\left(\binom{m}{n}\right)$ of the sets, where m is the number of integers in the interval.)

Proof of Theorem 1.1 Let S be a set of n real numbers from which no more than $g_\ell(n)$ numbers can be chosen to form a set containing no ℓ -term arithmetic progression, and let $u = g_\ell(n) + 1$. Then any set of u numbers from S contains an ℓ -term arithmetic progression. It follows that if A is the number of ℓ -term progressions in S , then

$$(1.4) \quad A \binom{n - \ell}{u - \ell} \geq \binom{n}{u}.$$

For, given an ℓ -term progression in S , there are $\binom{n - \ell}{u - \ell}$ sets of u elements of S that contain that progression, and if (1.4)

were false, there would be some set of u elements of S that contained no ℓ -term progression. Now, A does not exceed the number of 3-term progressions in S , and these number less than $n^2/4$. For, if the elements of S are $x_1 < x_2 < \dots < x_n$, then x_i is the middle term of at most $i-1$ 3-term progressions if $i \leq [(n+1)/2]$ and of at most $n-i$ progressions if $i > [(n+1)/2]$. Hence the number of 3-term progressions in S does not exceed $((n-1)/2)^2$ if n is odd and $(n/2)(n-2)/2$ if n is even.

Hence $\frac{n^2}{4} \binom{n-\ell}{u-\ell} > \binom{n}{u}$, so that

$$\frac{4}{n^2} < \frac{u! (n-\ell)_{u-\ell}}{(u-\ell)! (n)_u} = \frac{(u)_\ell}{(n)_\ell} < \left(\frac{u}{n}\right)^\ell,$$

and the theorem follows.

We might remark that a simple argument yields $g_3(n) > \sqrt{2n/3}$. Suppose that elements from a set S of n reals are chosen one at a time under the single restriction that the chosen numbers be progression-free. For each pair chosen, at most three other elements of S are excluded from further choice, so that after k numbers have been chosen, at most $k + 3\binom{k}{2}$ elements are excluded from further choice. If this is less than n , further choices can be made, so that the number k of elements selected can be at least large enough that $k + 3\binom{k}{2} \geq n$, whence $g_3(n) > \sqrt{2n/3}$. We have no improvement over (1.2) for $\ell > 3$. It may be that an estimate similar to (1.1) holds for $g_\ell(n)$.

In connection with the estimation of A in the above proof, it would be interesting to know whether a set of reals

contains the largest possible number of ℓ -term progressions when the set itself is an arithmetic progression. The argument above indicates that this is so in the case $\ell = 3$. If it were generally true, we could replace $n^2/4$ by $n^2/2(\ell-1)$ in the proof.

The proof of Theorem 1.2 consists mainly of the following lemma, which will be used also in the proof of Theorem 1.3.

Lemma 1.1 Let $b \geq 1$, and let n and w be positive integers with $w \leq n^b$. Then any set of w integers in $[1, n^b]$ contains a progression-free subset of cardinality at least

$$w/2^{3\sqrt{(2b \log n)/\log 2} + 3}.$$

Proof Consider the integers written in the binary scale, and the digits of an integer grouped into blocks as follows: counting from the right, the first block consists of the first digit, the second block of the next two digits, and so on, the last block filled out with zeros if it is incomplete. For example, 101110111 is grouped (0101)(110)(11)(1). Define the signature of an integer to be the sequence of first digits (from the right) of its blocks. The above number has signature 1,0,1,1, for example. Assume that $[n^b]$ has m blocks, so that

$$(1.5) \quad 2^{1+2+\dots+(m-1)} \leq n^b < 2^{1+2+\dots+m}.$$

An integer r in $[1, n^b]$ has a unique representation of the form

$$(1.6) \quad r = r_m 2^{\binom{m}{2}} + r_{m-1} 2^{\binom{m-1}{2}} + \dots + r_2 \cdot 2 + r_1,$$

where r_i is the number represented by the digits in the i -th block of r if such exists and otherwise $r_i = 0$.

Define the modulus of r :

$$M(r) = \sum_{i=1}^m r_i^2.$$

Suppose that the distinct integers r and s in $[1, n^b]$ have (I) the same signature, and (II) the same modulus. (I) implies that the representation of $(r+s)/2$ in the form (1.6) is

$$\sum_{i=1}^m \frac{r_i + s_i}{2} 2^{\binom{i}{2}}, \quad r_i + s_i \text{ being even for } i = 1, 2, \dots, m.$$

Hence, from (II) and Schwarz's inequality,

$$\begin{aligned} M((r+s)/2) &= \frac{1}{4} \{ 2 M(r) + 2 \sum_{i=1}^m r_i s_i \} \\ &\leq \frac{1}{4} \left\{ 2 M(r) + 2 \left(\sum_{i=1}^m r_i^2 \right)^{1/2} \left(\sum_{i=1}^m s_i^2 \right)^{1/2} \right\} = M(r). \end{aligned}$$

Equality holds here only if $r_i = cs_i$ for some $c > 0$, $i = 1, 2, \dots, m$, but then (II) would imply $c = 1$. Hence

$$M((r + s)/2) < M(r).$$

Therefore a set of integers which have the same signature and modulus does not contain the mean of any pair of its elements, and is thus progression-free.

The number of signatures among the integers $1, 2, \dots, [n^b]$

is at most 2^m , and the number of moduli is less than

$$2^{2m} + 2^{2(m-1)} + \dots + 2^2 = 4(2^{2m} - 1)/3.$$

Hence the number of different combinations of signatures and moduli corresponding to $1, 2, \dots, [n^b]$ is less than

$$\begin{aligned} 2^m \cdot 4(2^{2m} - 1)/3 &< \frac{4}{3} 2^{3m} < 2^{3m} + 1/2 \\ &\leq 2^{(3/2)\sqrt{8b \lg n} + 1} + 2 \\ &< 2^{3\sqrt{2b \lg n} + 3} \end{aligned} \quad \text{(by (1.5))}$$

where $\lg = \log_2$. Hence any set of w integers in $[1, n^b]$

contains at least $w/2^{3\sqrt{2b \lg n} + 3}$ elements having the same signature and modulus, and the lemma is proved.

Theorem 1.2 now follows by taking $b=a$ and $w=n$.

Before proving Theorem 1.3 we require three more lemmas.

By a P-set of intervals we mean a set of intervals

$$(1.7) \quad (u + (x_j - 1)v, u + x_j v], \quad j = 1, 2, \dots, m$$

where the x_j form a progression-free set of integers and u and $v \neq 0$ are real numbers. Considering the upper and lower halves of these intervals to be likewise half-open, we prove

Lemma 1.2 Given a P-set of intervals, the union of the upper halves contains no 3-term arithmetic progression with terms in different intervals. Similarly for the lower halves.

Proof Let (1.7) be the P-set of intervals. We can take $u = 0$ since translation does not affect the presence of arithmetic progressions in a set. One easily sees that the intervals

$$X_j = ((x_j - \frac{1}{2})v, x_j v], j = 1, 2, \dots, m$$

contain no 3-term progression with terms in different intervals. If two terms of a progression lie in one interval X_j , the third term cannot lie in another since the difference of the first two terms is less than the distance between intervals. If the three terms of a progression lay in X_{j1} , X_{j2} , and X_{j3} , then x_{j1} , x_{j2} , and x_{j3} would be in arithmetic progression.

Lemma 1.3 If a set of numbers has elements in each interval of a P-set of m intervals, then it contains a progression-free subset of at least $\lfloor (m+1)/2 \rfloor$ elements.

Lemma 1.3 is obvious from Lemma 1.2.

Lemma 1.4 Let $a \geq 3$. Then almost all sets of n integers from $(0, n^a]$ have elements occurring in at least $\lfloor n/3 \rfloor$ of the n^3 congruent intervals

$$(1.8) \quad (0, n^{a-3}], (n^{a-3}, 2n^{a-3}], \dots, ((n^3-1)n^{a-3}, n^a].$$

Proof Of the $\binom{[n^a]}{n}$ combinations of n integers from $(0, n^a]$, the number that have elements appearing in exactly j of the intervals (1.8) is at most

$$f(j) = \binom{n^3}{j} \sum_{b_1 + \dots + b_j = n} \binom{n^{a-3}+1}{b_1} \binom{n^{a-3}+1}{b_2} \dots \binom{n^{a-3}+1}{b_j}$$

where the sum is over all compositions of n into exactly j parts. If $b(k, n)$ is the number of these combinations having elements in fewer than k intervals, then

$$b(k, n) \leq \sum_{j=1}^{k-1} f(j) < (2n^{a-3})^n \sum_{j=1}^{k-1} n^{3j} \sum_{\substack{b_1 + \dots + b_j = n \\ b_i \geq 0}} \frac{1}{b_1! b_2! \dots b_j!}$$

$$< (2n^{a-3})^n \sum_{j=1}^{k-1} n^{3j} j^n / n!$$

(by the multinomial theorem)

$$< (2n^{a-3})^n n^{3k} k^{n+1}$$

Hence

$$b([n/3], n) < \frac{2^n n^{(a-3)n} n^n (n/3)^{n+1}}{n^{an-n}} \binom{n^a}{n} = o\left(\binom{[n^a]}{n}\right)$$

as $n \rightarrow \infty$.

Proof of Theorem 1.3 Let $a \geq 3$, and let S be a set of n integers in $[1, n^a]$ having elements in at least $[n/3]$ of the intervals (1.8). By Lemma 1.1, with $b = 3$, some

$[n/3]/2^{3\sqrt{6} \lg n + 3}$ of these $[n/3]$ intervals form a P-set of intervals, whence by Lemma 1.3, S contains a progression-free subset of cardinality at least

$$\frac{[n/3]}{2^{3\sqrt{6} \lg n + 4}} > n^{1-(3\sqrt{6} \lg n + 6)/\lg n}.$$

The theorem now follows from Lemma 1.4

We conclude this section with some calculations of $g_3(n)$ for small values of n . By definition,

$$(1.9) \quad g_\ell(n) \leq r_\ell(n),$$

and the values of these quantities for $\ell = 3$ and $n = 1, 2, \dots, 6$ are shown in the following table:

n	1	2	3	4	5	6
$r_3(n)$	1	2	2	3	4	4
$g_3(n)$	1	2	2	3	3	4.

The cases $n = 1, 2, 3$ are obvious. Since any set of 4 reals contains 3 elements not in progression, $g_3(4) \geq 3$, and since any 4-element subset of $\{0, 2, 3, 4, 6\}$ contains a 3-term progression, $g_3(5) \leq 3$. Hence $g_3(4) = g_3(5) = 3$. We now show that any set S of 6 reals contains a progression-free subset of 4 elements, so that $g_3(6) = 4$. (It is obvious that $r_3(6) = 4$, and that the other values of $r_3(n)$ shown are correct.)

With no loss of generality we can let the 2 least elements of S be 0 and 1. At least 3 of the remaining elements are different from 2; let these be $a < b < c$. Since $a \neq 2$, the numbers 0, 1, a are not in progression, and if $x \neq 2$ and $\{0, 1, a, x\}$ contains a progression then either $x = 2a - 1$ or $x = 2a$. Hence if $b \neq 2a - 1, 2a$, then $\{0, 1, a, b\}$ is progression-free, and similarly for c . The only remaining possibility is that $b = 2a - 1, c = 2a$, so that S contains 0, 1, $a, 2a - 1, 2a$. {Then 0, 1, $2a - 1, 2a$ } is a progression-free subset of S , for 2 is not in this set, and if 1, $2a - 1, 2a$ were in progression then $(2a - 1) - 1 = 2a - (2a - 1)$, or $2a - 1 = 2$.

As we see in the table, $g_3(5) < r_3(5)$. It would be interesting to know of other instances of strict inequality in (1.9).

§1.3 A related geometrical problem

Consider the numbers 0, 1, 2, ..., $\ell^n - 1$ written in the scale of ℓ , and regard the digits of each number, in their natural order, as the coordinates of a point in n -space. These points are all the lattice points in the cube $0 \leq x_i \leq \ell - 1$, $i = 1, 2, \dots, n$, and we shall call this set of ℓ^n points the ℓ^n - cube. We shall call a set of ℓ collinear points in the ℓ^n - cube a path, and let $M(\ell, n)$ be the cardinality of the largest path-free subset of the ℓ^n - cube.

When we consider (1.12) below, it will be evident that the numbers corresponding to the points of a path form an ℓ -term

arithmetic progression, and hence $r_\ell(\ell^n) \leq M(\ell, n)$. Therefore if $M(\ell, n) < \ell^n/k$ for some n , it would follow that $W(k, \ell) \leq \ell^n$. On the other hand, (1.1) provides a lower estimate for $M(\ell, n)$, and our object in this section is to find a larger estimate.

Theorem 1.4 Let $\ell \geq 3$ and $\varepsilon > 0$ be given. Then

$$(1.10) \quad M(\ell, n) > \left[\frac{\ell-1}{2} \right] \binom{n}{r_0} (\ell-1)^{n-r_0} (1-\varepsilon)$$

for n sufficiently large, where $r_0 = [(n+1)/\ell]$.

The cases $\ell = 1, 2$ are trivial since $M(1, n) = 0$ and $M(2, n) = 1$. For purposes of comparison with (1.1) we shall show that (1.10) implies

$$(1.11) \quad M(\ell, n) > \frac{1}{e^{2\sqrt{2\pi}}} \left[\frac{\ell-1}{2} \right] \frac{\ell^{n+1}}{\sqrt{n(\ell-1)}}$$

for n sufficiently large.

Proof of Theorem 1.4 If the coordinates of the ℓ points

of a path are $(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)})$, $i = 1, 2, \dots, \ell$,

then the column ℓ -tuples $(x_j^{(1)}, x_j^{(2)}, \dots, x_j^{(\ell)})'$, $j=1, 2, \dots, n$

are among the following $\ell + 2$ columns:

$$\begin{array}{cccccccc}
 & 0 & \ell-1 & 0 & 1 & 2 & \cdot & \cdot & \cdot & \ell-1 \\
 & 1 & \ell-2 & 0 & 1 & 2 & & & & \ell-1 \\
 & 2 & \ell-3 & 0 & 1 & 2 & & & & \ell-1 \\
 (1.12) & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \cdot \\
 & \ell-1 & 0 & 0 & 1 & 2 & & & & \ell-1 \cdot
 \end{array}$$

At least one of the first two columns must be included, for otherwise the ℓ points would be all the same. Therefore, at least one of the endpoints of a path contains more zeros among its coordinates than does any of the intermediate points. Hence a set of points that all have the same number of zeros among their coordinates will contain no path. The number of points of the ℓ^n -cube each having exactly r zeros among its coordinates is $\binom{n}{r}(\ell-1)^{n-r}$, and this quantity is maximized for $r = r_0$. This would yield (1.10) except for the factor $[(\ell-1)/2]$. This factor is obtained by observing that for $\ell \geq 5$, we can select more points for a path-free set than just those having r_0 zeros. In fact, for each $i = 0, 1, \dots, [(\ell-3)/2]$, let R_i be the set of all points in the ℓ^n -cube each having exactly $r_0 + i$'s among its coordinates, and let R^* be the set of all points each of which is in some one of these R_i but not in any other. We show that R^* contains no path, and that its cardinality is almost the sum of the cardinalities of the R_i .

Among the columns (1.12), a given path will have s columns $(0, 1, \dots, \ell-1)'$ and t columns $(\ell-1, \ell-2, \dots, 0)'$ for some non-negative integers s and t satisfying $0 < s + t \leq \ell - 1$.

We indicate these columns on the left in Table 1. Write $m = [(\ell-3)/2]$. On the right in the table are the frequencies with which $0, 1, \dots, m$ occur among the coordinates of the ℓ successive points of the path. In Table 1 we assume that ℓ is even. If ℓ is odd there will be one row $m+2, m+1, m+1; y_0, y_1, \dots, y_m$ in the centre of the table.

TABLE 1.

point #	s columns	t columns	Frequency of occurrence of 0 1 2 . . . m						
1	0	$\ell-1$	y_0^{+s}	y_1	y_2	.	.	.	y_m
2	1	$\ell-2$	y_0	y_1^{+s}	y_2	.	.	.	y_m
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
$m+1$	m	$m+3$	y_0	y_1	y_2	.	.	.	y_m^{+s}
$m+2$	$m+1$	$m+2$	y_0	y_1	y_2	.	.	.	y_m
$m+3$	$m+2$	$m+1$	y_0	y_1	y_2	.	.	.	y_m
$m+4$	$m+3$	m	y_0	y_1	y_2	.	.	.	y_m^{+t}
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
$\ell-1$	$\ell-2$	1	y_0	y_1^{+t}	y_2	.	.	.	y_m
ℓ	$\ell-1$	0	y_0^{+t}	y_1	y_2	.	.	.	y_m

For each point of the path lying in R^* , the corresponding row of the table contains exactly one frequency r_0 . If the $(m+2)$ -th point lies in R^* , then $y_i = r_0$ for some i and $y_j \neq r_0$ for $j \neq i$. Therefore, since $s + t > 0$, either there is no r_0 in the $(i+1)$ -th row of the table or there is no r_0 in the $(\ell-i)$ -th row. Hence in any path either the $(m+2)$ - th point lies outside R^* or some other point does.

Denoting the cardinality of a set S by $|S|$, we have for any $i, j = 0, 1, \dots, m, i \neq j$,

$$|R_i| = \binom{n}{r_0} (\ell-1)^{n-r_0}, \quad |R_i \cap R_j| = \frac{n! (\ell-2)^{n-2r_0}}{(n-2r_0)! (r_0!)^2},$$

and

$$(1.13) \quad |R^*| \geq (m+1) |R_i| - 2 \binom{m+1}{2} |R_i \cap R_j|.$$

For, a point that lies in h of the $m+1$ sets R_i is counted the following number of times on the right side of (1.13):

$$h - 2 \binom{h}{2} = 1 - (h-1)^2 = 1 \quad \text{if } h = 1, \\ \leq 0 \quad \text{if } h > 1.$$

If in (1.13) we included all of the $m+1$ terms that we might, the final result would not be affected since for large n each term after the first is negligible compared to the one preceding. We show that the second term is negligible compared to the first.

It is well known (see Robbins [16]) that for all positive integers h ,

$$(1.14) \quad e^{1/(12h+1)} < \frac{h!}{\sqrt{2\pi h} (h/e)^h} < e^{1/(12h)}.$$

The right side of (1.13) can be written

$$(m+1) \binom{n}{r_0} (\ell-1)^{n-r_0} \left\{ 1 - m \binom{n-r_0}{r_0} \frac{(\ell-2)^{n-2r_0}}{(\ell-1)^{n-r_0}} \right\},$$

and we apply (1.14) and the inequalities

$$(1.15) \quad (n+2)/\ell - 1 \leq r_0 \leq (n+1)/\ell$$

to show that

$$(1.16) \quad \binom{n-r_0}{r_0} \frac{(\ell-2)^{n-2r_0}}{(\ell-1)^{n-r_0}} = o\left(\frac{1}{\sqrt{n}}\right) \quad (n \rightarrow \infty).$$

By (1.14), since $\exp\{1/(12(n-r_0)) - 1/(12(n-2r_0)+1) - 1/(12r_0+1)\} < 1$, the left side of (1.16) is less than

$$\left(\frac{n-r_0}{2\pi(n-2r_0)r_0} \right)^{1/2} \cdot \frac{(n-r_0)^{n-r_0}}{(n-2r_0)^{n-2r_0} r_0^{r_0}} \cdot \frac{(\ell-2)^{n-2r_0}}{(\ell-1)^{n-r_0}}$$

(note that $r_0 > 0$ for $n \geq \ell-1$). Applying (1.15) we find that this quantity is less than

$$\left(\frac{\ell(\ell-1)}{2\pi n(\ell-2)} \right)^{1/2} \cdot \frac{\{1 + (\ell-2)/(n(\ell-1))\}^{n-r_0+1/2}}{\{1-2/(n(\ell-2))\}^{n-2r_0+1/2} \{1-(\ell-2)/n\}^{r_0+1/2}},$$

and then that the second factor here is less than e^2 provided n is large enough. Thus (1.16) follows.

Hence, if given $\varepsilon > 0$, we can obtain for sufficiently large n ,

$$|R^*| \geq (m+1) \binom{n}{r_0} (\ell-1)^{n-r_0} (1-\varepsilon),$$

and Theorem 1.4 follows.

To prove (1.11) we again use (1.14) and (1.15) and find that

$$\binom{n}{r_0} (\ell-1)^{n-r_0} > \frac{\ell^{n+1}}{\sqrt{2\pi n(\ell-1)}} \cdot \frac{\exp\{1/(12n+1) - 1/(12(n-r_0)) - 1/(12r_0)\}}{\{1 + (\ell-2)/(n(\ell-1))\}^{n-r_0+1/2} \{1+1/n\}^{r_0+1/2}}.$$

Now, since for positive h and x , $(1 + x/h)^h < e^x$, the second factor on the right exceeds

$$\exp \left\{ \frac{1}{12n+1} - \frac{1}{12(n-r_0)} - \frac{1}{12r_0} - \frac{\ell-2}{\ell-1} \left(1 - \frac{r_0^{-1/2}}{n} \right) - \frac{r_0^{+1/2}}{n} \right\},$$

and by (1.15) this can be shown to exceed

$$\exp \left\{ -1 + \frac{1}{\ell} - \frac{1}{n} \left[\frac{\ell-2}{\ell-1} \left(\frac{3}{2} - \frac{2}{\ell} \right) + \frac{1}{\ell} + \frac{1}{2} \right. \right. \\ \left. \left. + \frac{\ell}{12} \left(\frac{1}{\ell-1-1/n} + \frac{1}{1-(\ell-2)/n} - \frac{1}{\ell+\ell/(12n)} \right) \right] \right\}.$$

This in turn exceeds $e^{-3/2}$ if $n > \max\{2(\ell-2), (\ell+8)/2\}$.

Hence for sufficiently large n ,

$$M(\ell, n) > \left[\frac{\ell-1}{2} \right] \frac{\ell^{n+1}}{\sqrt{2\pi n(\ell-1)}} \cdot \frac{1-\varepsilon}{e^{3/2}}.$$

Finally, choosing $\varepsilon < 1-1/\sqrt{e}$ yields (1.11) for sufficiently large n .

Recalling Szekeres's conjecture that $r_\ell(n) = o(n)$, one might correspondingly conjecture that $M(\ell, n) = o(\ell^n)$ as $n \rightarrow \infty$. However, the only upper estimate we have for $M(\ell, n)$ is the rather trivial

$$(1.17) \quad M(\ell, n) \leq \ell^{n-1}(\ell-1).$$

To prove this we first observe that $M(\ell, 2) = \ell^2 - \ell$.

For, in a square lattice, it is necessary to remove at least one point from each horizontal and vertical path, and removing ℓ diagonal points suffices, to obtain a set free of paths.

(If ℓ is even, the points removed cannot all be from the same diagonal.) Since an ℓ^n -cube consists of ℓ disjoint ℓ^{n-1} -cubes,

$$M(\ell, n) \leq \ell M(\ell, n-1)$$

and (1.17) follows.

In Theorem 1.4 we considered ℓ fixed. On the other hand, fixing n , we have

$$(1.18) \quad M(\ell, n) \sim \ell^n \quad (\ell \rightarrow \infty).$$

For, since there are no ℓ collinear points in the $(\ell-1)^n$ -cube, $M(\ell, n) \geq (\ell-1)^n$. Hence from (1.17),

$$\left(\frac{\ell-1}{\ell} \right)^n \leq \frac{M(\ell, n)}{\ell^n} \leq \frac{\ell-1}{\ell},$$

implying (1.18).

CHAPTER II

SETS OF INTEGERS CONTAINING NO n TERMS IN GEOMETRIC PROGRESSION

§2.1 Introduction

In addition to proving the estimate (1.1), R. A. Rankin in the same paper [14] considered the problem of finding, for each integer $n \geq 3$, a sequence of positive integers containing no n -term geometric progression. He constructed such sets B_n having asymptotic density

$$A_n = \frac{1}{\zeta(n-1)} \prod_{k=1}^{\infty} \frac{\zeta\{(2n-3)^k\}}{\zeta\{(n-1)(2n-3)^k\}}.$$

For example, $A_3 \doteq .71975$, $A_4 \doteq .8626$, and $A_n \rightarrow 1$ as $n \rightarrow \infty$. The progressions under consideration have the form

$$k, kr, kr^2, \dots, kr^{n-1}$$

where k is a positive integer and r is a positive rational such that kr^{n-1} is integral. Note that the set of all positive integers not divisible by the $(n-1)$ -th power of any prime is free of such progressions and has density $1/\zeta(n-1)$, and that Rankin's set B_n has a higher density. B_n is the set of all integers N of the form

$$N = \prod_{i=1}^{\infty} p_i^{a_i}$$

where p_i is the i -th prime and $a_i \in C_n$ for $i=1, 2, \dots$, C_n

being the set of all non-negative integers which, when expressed in the scale of $2n-3$, contain no digit greater than $n-2$. C_n contains no n -term arithmetic progression, and it is easily seen that this implies that B_n contains no n -term geometric progression. Rankin wondered whether A_n is the largest density possible for a set free of n -term geometric progressions, but mentioned no upper bound for the possible density.

In this chapter we shall let $H(n)$, for $n \geq 3$, denote the class of all sequences of positive integers that contain no n -term geometric progression. In §2.2 we shall construct, for each $n \geq 4$, a member of $H(n)$ with density exceeding A_n . In fact, for each $n \geq 4$ there are several such members of $H(n)$. We find no member of $H(3)$ whose density exceeds A_3 . In §2.3 we shall find an upper estimate for the possible density of members of $H(n)$ for each $n \geq 3$. A table comparing some few of the densities we obtain with the corresponding upper estimates is provided at the end of §2.3.

We remark that, if we considered this problem for $n=2$, no member of $H(2)$ would contain more than one element since any two positive integers are in geometric progression. If we were to consider only geometric progressions with integral common ratio, we would have infinite sets free of 2-term progressions, for example, the set of primes. The question arises whether there is such a set with positive density. This is more than answered, in the negative, by Davenport and Erdős [5] who prove that any sequence $\{b_i\}$ with positive lower density con-

tains a subsequence $\{b_{i_k}\}$ such that $b_{i_k} | b_{i_{k+1}}$.

For any real x and set Q of positive integers we let $Q(x)$ denote the number of elements of Q that do not exceed x . If Q has asymptotic density, we shall denote it by $D(Q)$; that is, $D(Q) = \lim_{x \rightarrow \infty} Q(x)/x$ provided this limit exists.

If E is a set of non-negative integers, we shall write

$$Q(E) = \{N \mid N = \prod_{i=1}^{\infty} p_i^{a_i}, a_i \in E \text{ for } i=1,2,\dots\}$$

and call $Q(E)$ the set of integers developed from the exponent choice set E . For example $Q(\{0,1\})$ is the set of square-free numbers. We shall simplify the notation by writing $Q(\{a,b,\dots\}) = Q(a,b,\dots)$. If $0 \notin E$ then $Q(E)$ is empty, and we shall assume hereafter that any exponent choice set contains 0.

Before proceeding with our results we would like to remark on the considerations which led us to find denser members of $H(n)$. For $n=4$, for example, Rankin's exponent choice set is

$$C_4: 0, 1, 2, 5, 6, 7, 10, 11, 12, 25, \dots$$

Since there are few integers divisible by high powers of a prime, it seemed that we might obtain a denser sequence than $Q(C_4)$ by using a finite exponent choice set containing some smaller integers, and we observed that the set $E=\{0,1,2,4,5\}$

contains no 4-term arithmetic progression. A check of factor tables showed that $Q(E)$ has 91 elements below 100 while $Q(C_4)$ has only 88. Similar favorable comparisons for the respective numbers of elements below 200, 300, 400, and 500 led us to compute $D(Q(E))$, finding it exceeded $.88796 > A_4$. Roughly, it seems that the more small numbers in the exponent choice set the better. We find no improvement over A_3 because we cannot use the rule implied here on the set

$$C_3: 0, 1, 3, 4, 9, 10, 12, 13, 27, \dots$$

§2.2 Members of $H(n)$ with density exceeding A_n

For each $n \geq 4$ we shall find an exponent choice set E_n containing no n -term arithmetic progression and such that

$$D(Q(E_n)) > A_n = D(Q(C_n)).$$

In fact, these sets E_n will be used in proving the following theorem.

Theorem 2.1 (i) If n is prime, there exists $Q \in H(n)$ such that

$$(2.1) \quad D(Q) = \frac{\zeta(n)}{\zeta(n-1) \zeta\{(n-1)n\}}.$$

(ii) If n is odd and composite, there exists $Q \in H(n)$ such that

$$(2.2) \quad D(Q) > \frac{\zeta(n)}{\zeta(n-1)\zeta(hn)} - \left(\frac{1}{\zeta(hn-1)} - \frac{1}{\zeta(hn-h)} \right)$$

where h is the smallest prime divisor of n .

(iii) If n is even, there exists $Q \in H(n)$ such that

$$(2.3) \quad D(Q) > \frac{\zeta(n)}{\zeta(n-1)\zeta(2n)} - \left(\frac{1}{\zeta(2n-1)} - \frac{1}{\zeta(2n-2)} \right).$$

(iv) There exists $Q \in H(4)$ such that

$$(2.4) \quad D(Q) > .8952.$$

The estimate (2.4) is somewhat larger than that provided by (2.3) with $n=4$. We shall defer the proof that the respective densities exceed A_n to §2.21. We first prove part (iv) of the theorem.

Proof of (iv) The set

$$E_4': 0, 1, 2, 4, 5, 7, 8, 9$$

contains no 4-term arithmetic progression. We shall compute a lower estimate for $D(Q(E_4'))$. The set $Q(E_4')$ omits the positive integer m if and only if there is a prime p such that

$$p^3 \mid m \text{ and } p^4 \nmid m, \text{ or } p^6 \mid m \text{ and } p^7 \nmid m, \text{ or } p^{10} \mid m.$$

Given a prime p , the number of such m not exceeding x is

$$K(x, p) = [x/p^3] - [x/p^4] + [x/p^6] - [x/p^7] + [x/p^{10}],$$

and the density of the set of such numbers m is

$$K(p) = \lim_{x \rightarrow \infty} \frac{K(x, p)}{x} = \frac{1}{p^3} - \frac{1}{p^4} + \frac{1}{p^6} - \frac{1}{p^7} + \frac{1}{p^{10}}.$$

By the principle of inclusion and exclusion,

$$(2.5) \quad D(Q(E_4')) = 1 - \sum_p K(p) + \sum_{p < q} K(p) K(q) - \sum_{p < q < r} K(p) K(q) K(r) + \dots,$$

where the sums are respectively over all the tuples

(p) , (p, q) , (p, q, r) , ... of primes satisfying the indicated inequalities. Since $\sum_p K(p) < \sum_p 1/p^3 < 1$,

$$\begin{aligned} \sum_{p < q < r \dots} (K(p) K(q) K(r) \dots) &< \sum_p K(p) \cdot \sum_{q < r < \dots} (K(q) K(r) \dots) \\ (j \text{ primes}) & \hspace{15em} (j-1 \text{ primes}) \\ &< \sum_{q < r < \dots} (K(q) K(r) \dots), \\ & \hspace{10em} (j-1 \text{ primes}) \end{aligned}$$

and the terms of the alternating series (2.5) decrease in absolute value. Hence the series converges and we can obtain a lower estimate for (2.5) by using the first four terms. We

show this computation in the appendix, obtaining

$$D(Q(E_4')) > .8952.$$

Before proceeding with the remaining parts of the theorem we prove a lemma. If E is any exponent choice set, we can define a quantity $K_E(p)$ corresponding to $K(p)$ above, and if E contains 1, the series corresponding to (2.5) converges. Hence, for any exponent choice set E containing 1, $D(Q(E))$ exists and is the sum of this series. If $1 \notin E$, then $\sum_p K_E(p)$ diverges, for the first term of $K_E(p)$ is $1/p$; however, in this case $D(Q(E)) = 0$ (see remark #1 in §2.22). We thus have

Lemma 2.1 If E is any exponent choice set, then $D(Q(E))$ exists.

Proof of (i) If n is prime, then the set

$$\begin{array}{l} E_n: \quad 0, 1, 2, \dots, n-2, \\ \quad n, n+1, n+2, \dots, 2n-2, \\ \quad 2n, 2n+1, 2n+2, \dots, 3n-2, \\ \quad \dots \\ \quad (n-2)n, (n-2)n+1, (n-2)n+2, \dots, (n-1)n-2 \end{array}$$

contains no n -term arithmetic progression. For if E_n contained such progressions, one of them would have its first term among $0, 1, \dots, n-2$, and all of them would have common difference

less than n . However, if $0 \leq a \leq n-2$ and $1 \leq d \leq n-1$, some term of the progression

$$a, a+d, a+2d, \dots, a+(n-1)d$$

is congruent to -1 modulo n and hence is not in E_n . This is because $(d, n) = 1$, whence $0, d, 2d, \dots, (n-1)d$ form a complete residue system modulo n .

Now, with $s = \sigma + it$ (σ, t real),

$$\begin{aligned} \sum_{N \in Q(E_n)} \frac{1}{N^s} &= \prod_p \left(\sum_{a \in E_n} \frac{1}{p^{as}} \right) \\ &= \prod_p \frac{1 - 1/p^{(n-1)s}}{1 - 1/p^s} \cdot \frac{1 - 1/p^{(n-1)ns}}{1 - 1/p^{ns}} \\ (2.6) \quad &= \frac{\zeta(s) \zeta(ns)}{\zeta\{(n-1)s\} \zeta\{(n-1)ns\}}. \end{aligned}$$

We now employ the following lemma (see Ayoub [1, pp. 86-87], Theorem 6.12):

Lemma 2.2 If $f(s) = \sum_{N=1}^{\infty} \frac{a(N)}{N^s}$ and $\lim_{x \rightarrow \infty} \frac{\sum_{N \leq x} a(N)}{x} = A$,

then

$$(2.7) \quad \lim_{s \rightarrow 1} (s-1) f(s) = A.$$

Defining

$$a(N) = \begin{cases} 1 & \text{if } N \in Q(E_n) \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\sum_{N \in Q(E_n)} \frac{1}{N^s} = \sum_{N=1}^{\infty} \frac{a(N)}{N^s},$$

and with $Q = Q(E_n)$,

$$Q(x) = \sum_{N \leq x} a(N).$$

Lemma 2.1 assures us that $D(Q(E_n)) = \lim_{x \rightarrow \infty} Q(x)/x$ exists, and

by Lemma 2.2 we can find this limit from (2.7). It is the residue of (2.6) at the simple pole $s=1$. Thus

$$D(Q(E_n)) = \frac{\zeta(n)}{\zeta(n-1) \zeta\{(n-1)n\}},$$

and hence part (i) of theorem 2.1.

Note that we could adjoin integers to the above set E_n , still maintaining the property of being free of n -term progressions, and hence obtain a still denser member of $H(n)$. However, it is not clear whether one could express the density of any such enlarged set in a convenient form. In any case the density would be increased by only a very small amount. A

similar situation obtains in parts (ii) and (iii) of the theorem. We could, of course, express the density in a series of the form (2.5), but it requires a lengthy calculation to obtain estimates from such series, and it is impossible to compare the series in general with A_n .

Proof of (ii) Suppose that n is odd and composite, and that h is the smallest prime division of n . We shall show that the set

$$E_n: \begin{array}{ccccccc} & 0, & 1, & 2, & & & n-2, \\ & & & & & & \\ & n, & n+1, & n+2, & & & 2n-2, \\ & & & & & & \\ E_n: & 2n, & 2n+1, & 2n+2, & & & 3n-2, \\ & & & & & & \\ & (h-2)n, & (h-2)n+1, & (h-2)n+2, & \dots, & & (h-1)n-2, \\ & & & & & & \\ & (h-1)n, & (h-1)n+1, & (h-1)n+2, & \dots, & & hn-h-1 \end{array}$$

contains no n -term arithmetic progression. Consider any progression with first term a and common difference d such that

$$0 \leq a < a + d < a + 2d < \dots < a + (n-1)d \leq h(n-1)-1.$$

Evidently $d < h$. Hence $(d, n) = 1$. Therefore $a, a+d, \dots, a + (n-1)d$ form a complete residue system modulo n , whence this progression contains one of $n-1, 2n-1, \dots, (h-1)n-1$, and is not contained in E_n .

We cannot include the numbers $hn-h, hn-h+1, \dots, hn-2$ in E_n since each of these is the n -th term of an arithmetic

progression having difference h and first $n-1$ terms in E_n . If we ended the exponent choice set at $(h-1)n-2$, we would develop from it a member of $H(n)$ with density $\zeta(n)/(\zeta(n-1) \zeta\{(h-1)n\})$, but since the right side of (2.2) exceeds this, we use E_n as our exponent choice set. (See (2.15) for a proof that (2.2) is larger. The difference is indeed small, and even for the first case, with $n=9$ and $h=3$, the difference does not affect the fifth decimal place. However, the method we develop to obtain (2.2) will be necessary in the proof of part (iii) also.)

Using Lemma 2.2 we find that

$$(2.8) \quad D(Q(E_n \cup \{h_n-h, hn-h+1, \dots, hn-2\})) \\ = \zeta(n)/(\zeta(n-1) \zeta(hn)).$$

However, we must now make allowance for the necessary exclusion of $hn-h, hn-h+1, \dots, hn-2$.

Given any exponent choice set E and set of positive integers F , we shall denote by $Q(E \& F)$ the set of integers developed from the exponent choice set $E \cup F$ with always at least one element from each of E and F included among the exponents chosen (we always include the 0 from E). Then

$$(2.9) \quad Q(E) \cup Q(E \& F) = Q(E \cup F).$$

Now, by Lemma 2.1, $D(Q(E))$ and $D(Q(E \cup F))$ exist. Therefore

since the sets on the left side of (2.9) are disjoint,
 $D(Q(E \& F))$ exists and

$$(2.10) \quad D(Q(E)) + D(Q(E \& F)) = D(Q(E \cup F)).$$

Now by (2.10) we have

$$(2.11) \quad \begin{aligned} D(Q(E_n)) &= D(Q(E_n \cup \{h_{n-h}, h_{n-h+1}, \dots, h_{n-2}\})) \\ &\quad - D(Q(E_n \& \{h_{n-h}, h_{n-h+1}, \dots, h_{n-2}\})). \end{aligned}$$

Furthermore,

$$\begin{aligned} Q(E_n \& \{h_{n-h}, h_{n-h+1}, \dots, h_{n-2}\}) &\subset Q(\{0, 1, 2, \dots, h_{n-h-1}\} \& \\ &\quad \{h_{n-h}, h_{n-h+1}, \dots, h_{n-2}\}) \\ &= Q(0, 1, 2, \dots, h_{n-2}) - Q(0, 1, 2, \dots, h_{n-h-1}), \end{aligned}$$

where we have applied (2.9) again. Hence

$$\begin{aligned} D(Q(E_n \& \{h_{n-h}, h_{n-h+1}, \dots, h_{n-2}\})) &\leq D(Q(0, 1, 2, \dots, h_{n-2})) \\ &\quad - D(Q(0, 1, 2, \dots, h_{n-h-1})) \\ &= 1/\zeta(h_{n-1}) - 1/\zeta(h_{n-h}). \end{aligned}$$

Thence from (2.8) and (2.11),

$$D(Q(E_n)) \geq \zeta(n)/(\zeta(n-1)\zeta(h_n)) - [1/\zeta(h_{n-1}) - 1/\zeta(h_{n-h})]$$

and part (ii) of Theorem 2.1 follows.

Proof of (iii) If n is even, the set

$$E_n: 0, 1, 2, \dots, n-2, n, n+1, n+2, \dots, 2n-3$$

contains no n -term arithmetic progression. Now,

$$D(Q(E_n \cup \{2n-2\})) = \zeta(n)/(\zeta(n-1) \zeta(2n)),$$

but $E_n \cup \{2n-2\}$ contains the progression $0, 2, 4, \dots, 2n-2$.

By an argument like that in part (ii) above, using (2.9) and (2.10), we find that

$$D(Q(E_n)) \geq \zeta(n)/(\zeta(n-1) \zeta(2n)) - [1/\zeta(2n-1) - 1/\zeta(2n-2)]$$

and hence part (iii) of the theorem.

The proof of Theorem 2.1 is now complete.

A still denser member of $H(n)$ can be found in the case that n is even. Let ℓ be the smallest prime divisor of $n-1$. Then the set

$$E_n' = \{0, 1, 2, \dots, \ell(n-1)\} - \{n-1, 2(n-1), 3(n-1), \dots, (\ell-1)(n-1)\}$$

contains no n -term arithmetic progression. For, consider any progression with first term a and common difference d such that

$$(2.12) \quad 0 \leq a < a + d < a + 2d < \dots < a + (n-1)d \leq \ell(n-1).$$

Clearly $d \leq \ell$. We shall show that one of the terms of this progression is among the first $\ell-1$ multiples of $n-1$ and hence is not in E_n' . There are two cases:

Case that $(d, n-1) > 1$ Then $d=\ell$, since $1 < d \leq \ell$ and ℓ is the smallest prime divisor of $n-1$. Hence, by (2.12), $a=0$. Therefore, the progression in (2.12) is $0, \ell, 2\ell, \dots, (n-1)\ell$ and contains $\frac{n-1}{\ell} \cdot \ell = n-1$.

Case that $(d, n-1) = 1$ Then $d < \ell$. One term of $a + d, a + 2d, \dots, a + (n-1)d$ is a multiple of $n-1$ since $d, 2d, \dots, (n-1)d$ form a complete residue system modulo $n-1$. Since that term cannot be 0, it does not lie in E_n' unless it is $\ell(n-1)$. But if that were so, then a would be among $n-1, 2(n-1), \dots, (\ell-1)(n-1)$. For (2.12) would require $a + (n-1)d = \ell(n-1)$, whence $(n-1)|a$, and since $d < \ell$, $a \neq 0$.

Unfortunately, we are unable to use the simpler method (Lemma 2.2) to find $D(Q(E_n'))$. Consider for example the case $n = 4$ where

$$E_4 = \{0, 1, 2, 4, 5\} \quad \text{and} \quad E_4' = \{0, 1, 2, 4, 5, 7, 8, 9\}.$$

We find a lower estimate for $D(Q(E_4))$ in the proof of part (iii) of Theorem 2.1 by applying Lemma 2.2 to

$\sum_{N \in Q(E_4 \cup \{6\})} 1/N^s = \zeta(s) \zeta(4s)/(\zeta(3s) \zeta(8s))$. This yields

$D(Q(E_4 \cup \{6\})) = \zeta(4)/(\zeta(3) \zeta(8))$ and we can make an adjustment for the extra element 6 by using (2.9) and (2.10).

However, we have been unable to carry out this procedure with E_4' , where the Dirichlet series we would have to consider would be

$$(2.13) \quad \sum_{n \in Q(E)} 1/N^s$$

with E a suitable exponent choice set. In fact, to enable us to use Lemma 2.2 to obtain a better result than E_4 allows, E would require properties such that:

1. E contains E_4' , or $\{0, 1, 2, 4, 5, 7, 8\}$, or at least $\{0, 1, 2, 4, 5, 7\}$. (It would seem that to replace any of 1, 2, 4, 5, by any quantity of larger numbers would result in a set E with $D(Q(E)) < D(Q(E_4))$. Compare remark #2 in §2.22.)
2. One could find the residue of (2.13) at its simple pole $s=1$. (Lemmas 2.1 and 2.2 assure us that (2.13) has a simple pole at $s=1$, for since $1 \in E$, $A \neq 0$ in Lemma 2.2.)
3. One could make an adjustment for any elements in E that are not in E_4' and which form a 4-term arithmetic progression with other elements of E .

It is the third requirement that is difficult to meet.

For example, points 1 and 2 above are satisfied by

$E = \{0, 1, 2, \dots, 9\}$, but we would then have to find a

sharp upper estimate for $D(Q(E_4' \& \{3,6\}))$.

Recall that $D(Q(E_4')) > .8952$. By comparison the estimate (2.3) in the case $n=4$ is not larger than .88796 to five decimal places. Furthermore, estimating from above, we can show that $D(Q(E_4)) < .89093$. For, using (2.10),

$$\begin{aligned} D(Q(0, 1, 2, 4, 5)) &= D(Q(0, 1, 2, 3, 4, 5)) - D(Q(\{0,1,2,4,5\} \& \{3\})) \\ &\leq 1/\zeta(6) - D(Q(\{0, 1, 2\} \& \{3\})) \\ &= 1/\zeta(6) - [1/\zeta(4) - 1/\zeta(3)] < .89093. \end{aligned}$$

In closing this section we remark that the following theorem of Ikehara [9] can be useful in finding the density of some sets of the type $Q(E)$.

Ikehara's theorem Let $a(N) \geq 0$. If $f(s) = \sum_{n=1}^{\infty} \frac{a(N)}{N^s}$

converges absolutely for $\sigma > 1$ and f is analytic for $\sigma \geq 1$ except for a simple pole with residue A at $s=1$, then

$$\lim_{x \rightarrow \infty} \frac{\sum_{N \leq x} a(N)}{x} = A.$$

This theorem is, however, much deeper than Lemma 2.2. It does have the advantage that one need not prove the existence of the density before applying the theorem, but on the other hand, one must know that $\sum_{N \in Q(E)} 1/N^s$ has a simple pole at

$s = 1$. (Indeed, (I): the existence of a non-zero density, seems almost equivalent to (II): the existence of the pole. (I) implies (II) by Lemma 2.2. If we knew that the continuation onto $\sigma = 1$ of the function defined by the Dirichlet series were analytic for $\sigma = 1, t \neq 0$, then (II) would imply (I) by Ikehara's theorem. Also cf. remark #3 in §2.22.) The existence of the pole can be established in part (i) of Theorem 2.1 through knowledge of the ζ -function which appears in (2.6). However, for the less tractable exponent choice sets E_n of parts (ii) and (iii) of the theorem, the existence of this pole and the value of the density could not be found in the same way since we were unable to obtain $\sum_{N \in Q(E_n)} 1/N^s$ in any kind of closed form, let alone in terms of the ζ -function. Now, Lemmas 2.1 and 2.2 establish that since E_n contains 1, the Dirichlet series does have a simple pole at $s = 1$. But, if one must proceed so far with these Lemmas anyhow, Ikehara's theorem becomes superfluous. Furthermore, Lemma 2.1 was needed to prove the existence of density so that we could employ (2.10) in parts (ii) and (iii) of Theorem 2.1. Lastly we remark that Lemma 2.1 seems interesting aside from its applications above.

§2.21 Comparison of the densities

We shall show that

$$(2.14) \quad \zeta(n) / (\zeta(n-1) \cdot \zeta(2n)) - [1/\zeta(2n-1) - 1/\zeta(2n-2)] > A_n$$

for $n \geq 4$, and that

$$(2.15) \quad \zeta(n) / (\zeta(n-1) \cdot \zeta(hn)) - [1/\zeta(hn-1) - 1/\zeta(hn-h)] \\ > \zeta(n) / [\zeta(n-1) \cdot \zeta\{(h-1)n\}]$$

for $3 \leq h \leq n-2$. If n is an odd composite number and h is the smallest prime divisor of n , then $3 \leq h \leq \sqrt{n}$ and (2.15) will hold. Since the quantities $\zeta(n)/[\zeta(n-1)\zeta\{(n-1)n\}]$ of (2.1) and $\zeta(n)/[\zeta(n-1)\zeta\{(h-1)n\}]$ of (2.15) are each at least $\zeta(n)/(\zeta(n-1)\zeta(2n))$, it will follow from (2.14) and (2.15) that the densities in Theorem 2.1 (i), (ii), (iii), all exceed A_n .

We shall use the following lemma:

Lemma 2.3 For integers $a > 1$ and $b > 0$,

$$\zeta(a+b) < 1 + \frac{\zeta(a) - 1}{2^b}.$$

Proof Equivalently,

$$\zeta(a) - 1 > 2^b (\zeta(a+b) - 1),$$

and this follows from

$$\frac{1}{2^a} + \frac{1}{3^a} + \frac{1}{4^a} + \dots > \frac{2^b}{2^{a+b}} + \frac{2^b}{3^{a+b}} + \frac{2^b}{4^{a+b}} + \dots$$

To prove (2.14) we first show that

$$(2.16) \quad \prod_{k=1}^{\infty} \frac{\zeta\{(2n-3)^k\}}{\zeta\{(n-1)(2n-3)^k\}} < \frac{\zeta(2n-4)}{\zeta(2n^2)} \quad \text{for } n \geq 3$$

and then that

$$(2.17) \quad \zeta(2n-4)/\zeta(2n^2) < \zeta(n)/\zeta(2n) - \zeta(n-1)[1/\zeta(2n-1) - 1/\zeta(2n-2)]$$

for $n \geq 5$. This will imply (2.14) for $n \geq 5$. For $n=4$ we find using tables that the left side of (2.14) exceeds .88796, while $A_4 < .8627$.

We observe that

$$\zeta(2n^2) < \zeta\{(n-1)(2n-3)\} < \prod_{k=1}^{\infty} \zeta\{(n-1)(2n-3)^k\},$$

so that if we prove

$$(2.18) \quad \prod_{k=1}^{\infty} \zeta\{(2n-3)^k\} < \zeta(2n-4)$$

for $n \geq 3$ we shall have (2.16). Writing $m = 2n-3$, we shall prove

$$(2.19) \quad \zeta(m-1) > 1 + 2(\zeta(m)-1) > \prod_{k=1}^{\infty} \zeta(m^k) \quad \text{for } m \geq 3,$$

and this will yield (2.18) for $n \geq 3$. The first inequality of (2.19) is immediate from Lemma 2.3. Again from the lemma,

$$\prod_{k=1}^{\infty} \zeta(m^k) < \prod_{k=1}^{\infty} \left(1 + \frac{\zeta(m)-1}{2^{m^k-m}} \right).$$

Let us write $x = 2^m(\zeta(m) - 1)$ and show that

$$(2.20) \quad \prod_{k=1}^{\infty} \left(1 + \frac{x}{2^{m^k}} \right) < 1 + \frac{x}{2^{m-1}} \quad \text{for } m \geq 3,$$

whence will follow the second inequality of (2.19). Since

$$y(1-y/2) < \log(1+y) < y \quad (|y| < 1),$$

we find taking logarithms in (2.20) that

$$\sum_{k=1}^{\infty} \log \left(1 + \frac{x}{2^{m^k}} \right) < x \sum_{k=1}^{\infty} \frac{1}{2^{m^k}} < x \sum_{k=1}^{\infty} \frac{1}{2^{mk}} = x/(2^m-1)$$

and that

$$\log \left(1 + \frac{x}{2^{m-1}} \right) > \frac{x}{2^{m-1}} \left(1 - \frac{x}{2^m} \right).$$

Now,

$$\frac{x}{2^{m-1}} \left(1 - \frac{x}{2^m} \right) > \frac{x}{2^{m-1}}$$

if and only if

$$(2.21) \quad 2^{m-1} < (2^m-1)(2-\zeta(m)).$$

For $m \geq 3$, $2 - \zeta(m) > .7$, so that (2.21) follows if $2^m < (2^m - 1) \times (1.4)$, or if $0 < (.4)2^m - 1.4$. Since this does hold for $m \geq 3$, we have proved (2.16).

The inequality (2.17) is equivalent to

$$\frac{\zeta(2n-4)}{\zeta(2n^2)} + \zeta(n-1) \frac{\zeta(2n-2) - \zeta(2n-1)}{\zeta(2n-1)\zeta(2n-2)} < \frac{\zeta(n)}{\zeta(2n)}.$$

The second term on the left side here is less than

$\zeta(n-1)(\zeta(2n-2) - 1)/\zeta^2(2n)$. We replace that second term by this quantity, multiply through by $\zeta^2(2n) \zeta(2n^2)$, and show that the resulting inequality

$$\zeta^2(2n) \zeta(2n-4) + \zeta(2n^2) \zeta(n-1)(\zeta(2n-2)-1) < \zeta(n) \zeta(2n) \zeta(2n^2)$$

holds. Indeed, the left side here is less than

$$\zeta(2n) \zeta^2(2n-4) + \zeta(2n) \zeta(n-1)(\zeta(2n-2) - 1)$$

and we show that

$$(2.22) \quad \zeta^2(2n-4) + \zeta(n-1)(\zeta(2n-2) - 1) < \zeta(n) \zeta(2n^2).$$

By Lemma (2.3),

$$\zeta(2n-4) < 1 + \frac{\zeta(n) - 1}{2^{n-4}} \quad \text{and} \quad \zeta(2n-2)-1 < \frac{\zeta(n) - 1}{2^{n-2}},$$

and we shall find that

$$(2.23) \quad \left(1 + \frac{\zeta(n) - 1}{2^{n-4}}\right)^2 + \zeta(n-1) \frac{\zeta(n) - 1}{2^{n-2}} < \zeta(n).$$

This will imply (2.22) since $\zeta(2n^2) > 1$. Inequality (2.23) is equivalent to

$$\frac{\zeta(n) - 1}{2^{2n-8}} + \frac{1}{2^{n-5}} + \frac{\zeta(n-1)}{2^{n-2}} < 1.$$

Now for $n \geq 6$ the left side here does not exceed

$$\frac{\zeta(6) - 1}{2^4} + \frac{1}{2} + \frac{\zeta(5)}{2^4} < \frac{.02}{16} + \frac{1}{2} + \frac{1.04}{16} < 1.$$

Hence we have proved (2.17) for $n \geq 6$. For $n = 5$, (2.17) can be proved using tables.

We now prove (2.15). Equivalently we want to show that

$$(2.24) \quad \frac{\zeta(n)}{\zeta\{(h-1)n\}} + \zeta(n-1) \frac{\zeta(hn-h) - \zeta(hn-1)}{\zeta(hn-1) \zeta(hn-h)} < \frac{\zeta(n)}{\zeta(hn)}.$$

The second term on the left side here is less than

$\zeta(n-1)(\zeta(hn-h)-1)/\zeta^2(hn)$, and we shall show that (2.24) holds with this quantity place of the second term, or that

$$\zeta(n) \zeta^2(hn) + \zeta\{(h-1)n\} \zeta(n-1) (\zeta(hn-h)-1) < \zeta(n) \zeta\{(h-1)n\} \zeta(hn),$$

or equivalently

$$(2.25) \quad \zeta\{(h-1)n\} \zeta(n-1) (\zeta(hn-h)-1) < \zeta(n) \zeta(hn) (\zeta\{(h-1)n\} - \zeta(hn)).$$

By Lemma 2.3, $\zeta(hn-h) - 1 < (\zeta\{(h-1)n\} - 1)/2^{n-h}$ and also

$$\begin{aligned}\zeta\{(h-1)n\} - \zeta(hn) &> \zeta\{(h-1)n\} - 1 - (\zeta\{(h-1)n\} - 1)/2^n \\ &= (\zeta\{(h-1)n\} - 1)(2^n - 1)/2^n.\end{aligned}$$

Hence (2.25) will follow if

$$\zeta\{(h-1)n\} \zeta(n-1) < \zeta(n) \zeta(hn) (2^n - 1)/2^h.$$

Now, for $h \geq 3$ and $n \geq h + 2$ the right side here exceeds

$$(2^n - 1)/2^h \geq 4 - 1/2^h > 3$$

while the left side is less than 2. Hence (2.15) holds if

$$3 \leq h \leq n-2.$$

§2.22 Questions and remarks

1. We stated, preceding Lemma 2.1, that if an exponent choice set E does not contain 1, then $D(Q(E)) = 0$. This is so because $Q(E) \subset Q(0, 2, 3, 4, \dots)$, the set of squarefull numbers, and this set has density 0. We refer the reader to the solution by P. T. Bateman [2] of a problem proposed by D. J. Newman which shows that, if $Q = Q(0, 2, 3, 4, \dots)$, then

$$Q(x) = x^{1/2} \zeta(3/2)/\zeta(3) + x^{1/3} \zeta(2/3)/\zeta(2) + O(x^{1/5}).$$

Is there a way to prove the remainder of Lemma 2.1 without

using the inclusion-exclusion argument?

2. Is it true that if E is an exponent choice set and $m = \max E$ then $D(Q(E)) > D(Q([E - \{m\}] \cup \{m+1, m+2, \dots\}))$? Compare the "rule" mentioned at the end of §2.1.

3. For any set Q of positive integers let $f(s) = \sum_{N \in Q} 1/N^s$. If $D(Q) = 0$ then by Lemma 2.2 f has no pole at $s = 1$. Conversely, suppose that f has no pole at $s = 1$. Does it follow that $D(Q) = 0$? If we know that $D(Q)$ exists, then it is 0, for if it exceeded 0, Lemma 2.2 would imply that f has a pole at $s = 1$. We might ask: Are there sets Q such that $\limsup Q(x)/x \neq \liminf Q(x)/x$ and f has no pole at $s = 1$?

Is there a simple rule by means of which one can find an asymptotic estimate for $Q(x)$ from $f(s)$? For example, if Q is the set of k -th powers, then $f(s) = \zeta(ks)$ has the simple pole $s = 1/k$ and $Q(x) \sim x^{1/k}$. If Q is the set of square-full numbers then $f(s) = \zeta(2s) \zeta(3s)/\zeta(6s)$ has its largest pole at $s = 1/2$, and $Q(x) \sim x^{1/2} \zeta(3/2)/\zeta(3)$.

Is it generally the case that if the largest simple pole of f is r ($0 < r < 1$) then $Q(x) \sim c x^r$? One can prove the converse:

$$Q(x) \sim c x^r \quad (x \rightarrow \infty) \text{ implies } f(s) \sim \frac{cr}{s-r} \quad (s \rightarrow r)$$

by following the method of proof of our Lemma 2.2 as given in Ayoub [1, p. 87].

§ 2.3 Upper estimates for the density of members of $H(n)$

We define

$$M_n^* = \sup \{ \limsup_{x \rightarrow \infty} Q(x)/x \mid Q \in H(n) \}.$$

Then the densities of §§2.1 and 2.2 do not exceed M_n^* for $n \geq 3$.

So far in this chapter we have considered only infinite sets of integers containing no n -term geometric progression. Now, for any real x , let us denote by $q_n(x)$ the maximum number of integers that can be selected from $[1, x]$ so as not to contain any n -term geometric progression, and define

$$m_n^* = \limsup_{x \rightarrow \infty} q_n(x)/x.$$

For every Q in $H(n)$ and every x , $Q(x) \leq q_n(x)$.

Hence

$$M_n^* \leq m_n^*,$$

so that an upper estimate for m_n^* affords one for M_n^* .

We shall prove the following theorem.

Theorem 2.2 (i) For every $n \geq 3$,

$$(2.26) \quad M_n^* \leq 1 - 1/(2^n - 1).$$

(ii) In the case $n=3$, the better estimate

$$(2.27) \quad M_3^* < .8339$$

holds.

The proof of Theorem 2.2 depends on the following Theorem 2.3, which deals with geometric progressions of integral ratio r . Let I denote the set of positive integers. For any integer $r > 1$ let $R = R(n, r)$ denote the set of geometric progressions in I of n terms and ratio r , and let $H(n, r)$ denote the class of all sequences in I that contain no progression of R . Further, let $H_1(n, r)$ be the class of all sequences $Q \in H(n, r)$ for which $\lim_{x \rightarrow \infty} Q(x)/x$ exists. We define

$$M_{n,r}^* = \sup\{\limsup_{x \rightarrow \infty} Q(x)/x \mid Q \in H(n, r)\},$$

$$M_{n,r} = \sup\{\lim_{x \rightarrow \infty} Q(x)/x \mid Q \in H_1(n, r)\}.$$

Theorem 2.3 (i) No integer appears in more than n progressions of R .

(ii) There is exactly one subset of I with the property: Each element of the set appears in n progressions of R and each progression of R contains exactly one element of the set.

(iii) Let S be the set in (ii). If $T \subset I$ and $I - T \in H(n, r)$, then $T(x) \geq S(x)$ for every x .

(iv) $I - S \in H_1(n, r)$ and $\lim_{x \rightarrow \infty} S(x)/x = (r-1)/(r^n-1)$.

(v) $M_{n,r} = M_{n,r}^* = 1 - (r-1)/(r^n-1)$.

If analogously to $M_{n,r}$ one defined

$$M_n = \sup \{ \lim_{x \rightarrow \infty} Q(x)/x \mid Q \in H_1(n) \},$$

where $H_1(n)$ is the class of all sequences Q in $H(n)$ for which $\lim Q(x)/x$ exists, one might expect that similarly $M_n = M_n^*$. Perhaps this would be so if one considered only geometric progressions with integral ratio, but it seems doubtful in the general case.

Proof of Theorem 2.3 Let us separate R into families F_k of progressions:

$$\begin{array}{ccccccc} k, & kr & , & kr^2 & , & . & . & . & , & kr^{n-1} & ; \\ & & & & & & & & & & \\ kr & , & kr^2 & , & kr^3 & , & . & . & . & , & kr^n & ; \end{array}$$

F_k :

$$\begin{array}{ccccccc} & & & & & & . \\ & & & & & & . \\ & & & & & & . \\ kr^{n-1}, & kr^n & , & kr^{n+1}, & . & . & . & , & kr^{2n-2}; \\ kr^n & , & kr^{n+1}, & kr^{n+2}, & . & . & . & , & kr^{2n-1}; \end{array}$$

where $k \in I$, $r \nmid k$. Clearly $\bigcup_{\substack{k=1 \\ r \nmid k}}^{\infty} F_k = R$, and no integer appears in more than n progressions of one family. Furthermore,

if V_k denotes the set of all integers appearing in the progressions of F_k , then the sets V_k are pairwise disjoint. For if $kr^u = \ell r^v$ and $k \neq \ell$, then $u \neq v$ and either $r|k$ or $r|\ell$. (i) follows.

The integers $kr^{n-1}, kr^{2n-1}, kr^{3n-1}, \dots$ each appear in exactly n of the progressions of F_k , and each progression of F_k contains exactly one of them; it is clear that this is the only set of integers with this property. Since the V_k are pairwise disjoint, the set

$$S = \bigcup_{\substack{k=1 \\ r \nmid k}}^{\infty} \{kr^{n-1}, kr^{2n-1}, kr^{3n-1}, \dots\}$$

has the property required in (ii). It is clear that $I - S \in H(n, r)$.

Proceeding to (iii), we observe that if each F_k is separated into blocks of n progressions each, starting with the first member of the family, then in order that $I - T \in H(n, r)$, T must contain at least one integer from each block of each family. Since S contains exactly one integer from each block, $T(x) \geq S(x)$ for every x .

The number of integers

$$(2.28) \quad r^{in-1}, 2r^{in-1}, \dots, (r-1)r^{in-1}, (r+1)r^{in-1}, \dots$$

not exceeding x is

$$a_i = \left[\frac{x - r^{in-1}}{r^{in}} + 1 \right] + \left[\frac{x - 2r^{in-1}}{r^{in}} + 1 \right] + \dots + \left[\frac{x - (r-1)r^{in-1}}{r^{in}} + 1 \right]$$

$$= \left[\frac{x}{r^{in}} + \frac{r-1}{r} \right] + \left[\frac{x}{r^{in}} + \frac{r-2}{r} \right] + \dots + \left[\frac{x}{r^{in}} + \frac{1}{r} \right]$$

provided $1 \leq i \leq m = \left\lfloor \frac{\log_r x + 1}{n} \right\rfloor$, while if $i > m$, the integers (2.28) all exceed x . Hence

$$S(x) = \sum_{i=1}^m a_i = \sum_{i=1}^m \frac{(r-1)x}{r^{in}} + O(\log x),$$

so that S has density

$$\lim_{x \rightarrow \infty} \frac{S(x)}{x} = \sum_{i=1}^{\infty} \frac{r-1}{r^{in}} = \frac{r-1}{r^n - 1},$$

and we have proved (iv).

From (iv), $M_{n,r} \geq 1 - (r-1)/(r^n - 1)$. On the other hand, by (iii), if $U \in H(n, r)$, then $U(x) \leq [I-S](x)$ for every x , so that $M_{n,r}^* \leq 1 - (r-1)/(r^n - 1)$. Since $M_{n,r} \leq M_{n,r}^*$ by definition, (v) follows.

Proof of Theorem 2.2 Part (i) of the theorem follows immediately from the observation that $M_n^* \leq M_{n,r}^*$ for any r . We choose $r = 2$ since $M_{n,r}^*$ is smallest for that value of r .

Now let us denote by I_x the set of positive integers not exceeding x , by S_r the set S of the preceding proof, and by $P(n, r, x)$ the set of geometric progressions of n terms and common ratio r in I_x . From Theorem 2.3 part (iii) it

follows that some optimal set of $S_2(x)$ elements in I_x contains some term from each progression of $P(n, 2, x)$ and that no smaller set does so. We can say that the removal of such an optimal set of $S_2(x)$ elements from I_x eliminates the progressions of $P(n, 2, x)$ from I_x , and that the removal of a smaller set will not do so.

Note that we could also have proved part (i) of the theorem as follows: To eliminate all n -term geometric progressions from I_x we must eliminate at least those of $P(n, 2, x)$. Therefore $q_n(x) \leq x - S_2(x)$ for any x , whence $m_n^* \leq 1 - 1/(2^n - 1)$.

Turning to part (ii) of Theorem 2.2, we shall show that more than $S_2(x)$ numbers must be removed from I_x in order to eliminate additionally certain of the progressions of $P(3, 3, x)$. When the meaning is clear from the context, we shall shorten the notation " $P(3, r, x)$ " to " $P(r, x)$ ".

The progressions of $P(r, x)$ have the form k, kr, kr^2 where $1 \leq k \leq x/r^2$, and their terms fall into disjoint families of integers as in the proof of Theorem 2.3. For progressions $s, 3s, 9s$ of $P(3, x)$ we shall consider in which families of $P(2, x)$ these terms appear and see that for some values of s , the minimal number of choices required to eliminate these families of $P(2, x)$ either is insufficient to eliminate $s, 3s, 9s$, or if sufficient, still fails to eliminate both $s, 3s, 9s$ and some other progression of $P(3, x)$

whose terms appear only in those same families of $P(2,x)$.

For example let $x = 25$. The progressions of $P(2,25)$ and $P(3,25)$, arranged in families, are respectively

1, 2, 4 3, 6, 12 5, 10, 20 and 1, 3, 9 2, 6, 18
2, 4, 8 6, 12, 24
4, 8, 16

The first two families of $P(2, 25)$ are the ones that contain terms of the progressions in $P(3, 25)$. Any optimal selection of $S_2(25) = 3$ elements of I_{25} for the purpose of eliminating these families of $P(2, 25)$ would include 4 and one of 6 and 12. These choices leave the progression 1, 3, 9 intact, and hence one additional choice is required to eliminate this member of $P(3, 25)$.

In the following we assume that x is large enough that the progressions $s, 3s, 9s$ of $P(3,x)$ with which we shall deal do indeed exist. By $2||s$ we shall mean that 2 divides s but no higher power of 2 does so. We shall show that, for each integer s satisfying

$$(2.29 \text{ a}) \quad x/12 < s \leq x/9 \quad \text{and either } 2 \nmid s \text{ or } 2||s$$

or

$$(2.29 \text{ b}) \quad 2x/27 < s \leq x/12 \quad \text{and } 2||s,$$

one choice, in addition to the optimal $S_2(x)$ choices required for the elimination of $P(2, x)$, is required for the elimination of certain progressions associated with s in $P(3, x)$. We shall then show that the choices associated with one value of s do not coincide with those associated with any other, so that as many additional choices are required as there are integers s satisfying (2.29).

First let $x/12 < s \leq x/9$. Then $3s > x/4$, so that the only term of the progression $s, 3s, 9s$ that appears as the first term of a progression in $P(2, x)$ is s . We consider two cases.

Case I : s is odd In this case $3s$ and $9s$ are odd and appear in no progression of $P(2, x)$ since the second and third terms of these progressions are even. Hence the only family of progressions of $P(2, x)$ that contain any of $s, 3s, 9s$ is

$$\begin{array}{ccc} s, & 2s, & 4s; \\ 2s, & 4s & 8s; \end{array}$$

($16s > x$). To eliminate this family and the progression $s, 3s, 9s$, two choices from I_x are required, while the family itself requires only one choice.

Case II: $2 \parallel s$ The families of $P(2, x)$ that contain any of $s, 3s, 9s$ are

$$\begin{array}{ccc} s/2, & s, & 2s; & \text{and} & 3s/2, & 3s, & 6s; \\ s, & 2s, & 4s; \\ 2s, & 4s, & 8s; \end{array}$$

$(12s > x)$. These are also the only families of $P(2, x)$ that contain any term of the progression $s/2, 3s/2, 9s/2$. Any optimal selection from I_x for the elimination of the above families of $P(2, x)$ will leave one or both of these ratio 3 progressions intact, and one additional choice is required.

While further cases $2^2 || s$, $2^3 || s$, and so on, might be considered, the improvement they would afford to the estimate (2.27), if any, is too small to warrant the more complex argument. In some cases, the "additional" choices would coincide with those associated with integers s in (2.29 b). We turn now to the latter.

Given an integer s satisfying (2.29 b), the families of $P(2, x)$ that contain any of $s, 3s, 9s$ are

$$\begin{array}{ll} s/2, s, 2s; & \text{and } 3s/2, 3s, 6s; \\ s, 2s, 4s; & 3s, 6s, 12s; \\ 2s, 4s, 8s; & \end{array}$$

($9s$ could appear only as the first or second term of a ratio 2 progression, but the third term of such a progression would be at least $18s > x$). The argument finishes as in Case II above.

We remark that for odd integers s satisfying $2x/27 < s \leq x/12$, not considered above, an additional choice is required only if $3^2 || s$ or $3^\alpha || s$ for some $\alpha > 2$. However, the small reduction the argument might afford the estimate does not warrant the difficulty in proving whether such a choice would not coincide with some other choice. In fact, this reduction would be at most $\frac{2}{3} \cdot \frac{1}{9} \cdot \frac{1}{108} \doteq .00068$ and would not affect the third decimal place in (2.27).

We now show that the possible choices associated with one value of s satisfying (2.29) do not coincide with those associated with another. Indeed, if

$$(2.30) \quad 2x/27 < s \leq x/9,$$

then the choices associated with s in the arguments above are among

$$s/2, s, 2s, 4s, 8s, 3s/2, 3s, 6s, 12s, 9s/2, 9s \quad \text{if } 2 \mid s$$

and among

$$s, 2s, 4s, 8s, 3s, 9s \quad \text{if } 2 \nmid s.$$

Considering the exponents of the powers of 2 occurring as factors of choices associated with distinct s_1 and s_2 satisfying (2.30), the only possible equalities between choices associated with s_1 and those associated with s_2 are equivalent to equating s_1 to one of $3s_2, 9s_2, s_2/3, s_2/9$ unless s_1 and s_2 have opposite parity. If, say, $2 \mid s_1$ and $2 \nmid s_2$, the possible equalities are equivalent to equating s_1 to one of $2s_2/9, 2s_2/3, 2s_2, 18s_2$. But none of these equations would be consistent with the inequalities

$$2/3 < s_1/s_2 < 3/2$$

which hold for any s_1 and s_2 satisfying (2.30).

To count the number N of integers s satisfying (2.29), we note that the number of odd integers in an interval $(a, b]$ is

at least $([b-a] - 1)/2$, and that the number of integers s in $(a, b]$ such that $2 || s$ is at least $([b-a] - 1)/4 - 1/2$. Hence

$$\begin{aligned} N &\geq ([x/9 - x/12] - 1)/2 + ([x/9 - 2x/27] - 1)/4 - 1/2 \\ &= [x/36]/2 + [x/27]/4 - 5/4. \end{aligned}$$

Therefore

$$\begin{aligned} q_3(x) &\leq x - S_2(x) - [x/36]/2 - [x/27]/4 + 5/4 \\ &\sim x - x/7 - x/72 - x/108 \quad (x \rightarrow \infty) \\ &< .833995 x. \end{aligned}$$

Hence $m_3^* < .8339$ and (2.27) follows. The proof of Theorem 2.2 is now complete.

Let D_n be the density of the most dense member of $H(n)$ discussed in §§2.1 and 2.2. We compare D_n with our estimates for M_n^* for some few values of n :

n	$D_n \geq$	$M_n^* <$
3	$A_3 = .7197\dots$	$.8339$
4	$.8952$	$14/15 = .933\bar{3}$
5	$.9580$	$30/31 = .9677\dots$
8	$.9957$	$254/255 = .9960\dots$

Evidently there is little difference between D_n and M_n^*

for $n \geq 8$. It would be interesting to know whether A_3 can be improved upon. Certainly the estimate for M_3^* could be lowered by considering geometric progressions of ratios other than 2 and 3. In particular, perhaps the geometric progressions of non-integral ratio account for much of the difference between A_3 and the above estimate for M_3^* .

CHAPTER III

ON BASES FOR SETS OF INTEGERS

§3.1 Introduction

Let $h \geq 1$ and $n \geq 0$ be integers. A set $A: 0 = a_1 < a_2 < \dots < a_k \leq n$ of real numbers is called an h -basis for n if each of $0, 1, 2, \dots, n$ can be represented as the sum of h summands a_i from A , repetitions being allowed. Often in the literature an h -basis for n is called a basis of order h for n . By a minimal h -basis for n is meant an h -basis for n with the smallest possible number of elements, and $k_h(n)$ is used to denote the number of elements in such a basis. We shall often refer to an h -basis for n simply as an h -basis, or a basis for n , or a basis, when the meaning is clear from the context. The case $h = 1$ is trivial since then necessarily A contains $0, 1, 2, \dots, n$. In the following we assume $h \geq 2$.

Some authors consider only bases whose elements are integers. See for example Rohrbach [17] and Stöhr [20, p.46]. However, in [12], Moser observed that the result he obtains there is independent of whether the basis elements are integers. Since our results in this chapter have this property also, we use the definition above.

If the elements of an h -basis for n do not exceed n/h ,

we call the basis a restricted h-basis for n. Rohrbach (op. cit.) seems to have been the first to have considered this restricted case. If the basis-elements were specifically integers, we would require $[(n+h-1)/h]$ in place of n/h above, but the results we obtain in the following are the same either way.

In the following, we let A be any h -basis for n with k elements. This includes the possibility that $k = k_h(n)$. The problem of estimating $k_h(n)$, which we consider in this chapter, seems first to have been proposed by I. Schur, for the case $h = 2$ (see Rohrbach, op. cit., pp. 2,3). Rohrbach, by constructing h -bases, obtained the estimate

$$(3.1) \quad k_h(n) < h n^{1/h}.$$

(In the case $h = 2$, his basis is even restricted.) On the other hand, since we can form $\binom{k+h-1}{h}$ h -combinations with repetition of the k elements of A , and therefore that many sums, we have

$$(3.2) \quad \binom{k+h-1}{h} \geq n+1.$$

Because $(k+h-1)^h/h! > \binom{k+h-1}{h}$, (3.2) yields for any $\epsilon > 0$

$$(3.3) \quad \frac{k^h}{h!} (1+\epsilon) > n$$

provided n is sufficiently large. In the case $h = 2$,

Rohrbach (op. cit., p. 9) conjectured that $k_2^2(n) \sim 4n$, but obtained only a small improvement over (3.3), namely

$$(3.4 \text{ a}) \quad \frac{k^2}{2} (1 - .0016) > n.$$

Further improvements in the lower estimate were found by Moser [12] and the author [15], the latter result being

$$(3.4 \text{ b}) \quad \frac{k^2}{2} (1 - .0269) > n$$

for n large. In the case $h \geq 3$ the only improvement on (3.3) is that of Moser and the author [13] who showed that for any $\epsilon > 0$,

$$(3.5) \quad \frac{k^h}{h!} \left\{ 1 + \epsilon - \left(\frac{\cos(\pi/h)}{2 + \cos(\pi/h)} \right)^h \right\} > n$$

provided n is sufficiently large.

If A is a restricted h -basis for n , the lower estimates for k are larger, as one might expect. To date the best lower estimates have been, in the case $h = 2$,

$$(3.6) \quad \frac{k^2}{2} (1 - .1329) > n$$

if n is large (the author [15]), and in the case $h \geq 3$,

$$(3.7) \quad \frac{k^h}{h!} \{1 + \epsilon - [\cos(\pi/h)]^h\} > n$$

if n is large (Moser and the author [13]). Regarding upper

estimates, (3.1) holds if $h = 2$, but we know of no construction of a restricted h -basis A with $h \geq 3$ whose elements number less than $hn^{1/h}$.

Rohrbach's method in obtaining (3.4 a) consisted of dividing $[0, n]$ into eight intervals and considering the consequences of various distributions of the basis-elements into these intervals. Stöhr [20, p. 43] felt that it might be possible to use this method to sharpen the lower estimate (3.3) in the cases $h > 2$, but also stated that the calculations would become extensive and impossible to see through. In the paper [13] by Moser and the author, however, we obtain (3.5) by means of a quite brief argument. Stöhr's concern with such results was with their application to a problem on bases for the set of all non-negative integers, which we now define.

A set B of non-negative numbers is called an h -basis for $\mathbb{Z}$, the set of non-negative integers, if every element of $\mathbb{Z}$ can be represented as the sum of h summands from B , repetitions being allowed. Again when the context is clear, we may refer to these bases simply as " h -bases".

The idea of bases, and their name, originated in Germany in the early 1930's; Stöhr (op. cit., p. 41) attributes it to Schnirelmann, whose notion was, however, somewhat different from that of the present day. One finds the modern idea of basis first in the 1937 paper of Rohrbach (op. cit., p. 2). One could generalize the idea as follows: If S and T are sets of

numbers, and every element of T can be represented as the sum of h summands from S , then S is an h -basis for T . Under this definition, we can say that the basis problems that have been mainly of interest concern the cases with $T = \{0, 1, \dots, n\}$ and $T = \mathbb{Z}$. In these cases, German mathematicians have of late used the words "Abschnittsbasis" and "Basis" respectively, Stöhr coining the former of these (op. cit., p. 46). We find it more convenient to use the terminology of our definitions above than to employ a special word for the finite bases. Indeed, our terminology is the same that Rohrbach employed in the work cited above. The kind of basis that we are discussing is always made clear, and if not explicitly, then from the context.

In this chapter we improve upon the above lower estimates (3.4b) through (3.7). In the cases corresponding to (3.4 b), (3.5), and (3.6) we employ a refinement of the methods that were used to obtain those results. However, in the restricted case with $h \geq 3$, we obtain a considerable improvement over (3.7) by means of a very simple combinatorial argument. In §3.2 we consider the case that A is unrestricted. Although our method is basically the same as that in [15] and [13], where (3.4b) and (3.5) were obtained, this is the first time that all cases $h \geq 2$ are handled by a single argument. The argument in [13], which yielded (3.5), yields no improvement over (3.3) in the case $h = 2$. In §3.3 we consider the case that A is a restricted h -basis for n . Finally in §3.4 we discuss

applications of our results to some problems on h -bases for Z .

§3.2 The unrestricted case

It is convenient to denote the expression $\frac{\cos(\pi/h)}{2 + \cos(\pi/h)}$ found in (3.5) by $s(h)$. We shall prove the following theorem.

Theorem 3.1 (i) The case $h = 2$: For n sufficiently large,

$$(3.8) \quad \frac{k^2}{2} (1 - .0306) > n.$$

(ii) The case $h = 3$: For n sufficiently large,

$$(3.9) \quad \frac{k^3}{3!} (1 - .0221) > n.$$

(iii) The case $h = 4$: For n sufficiently large,

$$(3.10) \quad \frac{k^4}{4!} (1 - .01154) > n.$$

(iv) For each $h \geq 5$,

$$(3.11) \quad \frac{k^h}{h!} \{1 - [1.02 s(h)]^h\} > n$$

provided $n > n_0(h)$.

(v) For each $h \geq 8$,

$$(3.12) \quad \frac{k^h}{h!} \{1 - [1.1 s(h)]^h\} > n$$

provided $n > n_1(h)$.

To compare (3.9) and (3.10) with (3.5), we note that

$$s^3(3) = 1/5^3 = .008 < .0221$$

and

$$s^4(4) = 1/(2\sqrt{2} + 1)^4 < .00465 < .01154.$$

Proceeding to the proof of the theorem we consider the generating function

$$f(z) = \sum_{j=1}^k z^{a_j}$$

and let

$$(3.13) \quad g(z) = \frac{f^h(z)}{h!} + f^{h-2}(z) f(z^2).$$

The coefficient of z^j in $g(z)$ will be greater than or equal to the number of representations of j as the sum of h summands from A with order not counted. If we now define $\delta(j)$ by

$$(3.14) \quad g(z) = z + z^2 + \dots + z^n + \sum_{j=0}^{ha_k} \delta(j) z^j,$$

then $\delta(j) \geq 0$ for all j since A is an h -basis for n .

Putting $z = 1$ in (3.14) yields

$$(3.15) \quad \frac{k^h}{h!} + k^{h-1} = n + \sum \delta(j).$$

Note that (3.15) implies (3.3). The results of the theorem are obtained, as were the earlier results (3.4 b) through (3.7), by showing that $\sum \delta(j)$ is large.

To this end we set $\omega = e^{2\pi i/n}$, so that
 $\omega^t + \omega^{2t} + \omega^{3t} + \dots + \omega^{nt} = 0$ if and only if $n \nmid t$. Hence
 from (3.14) we obtain

$$g(\omega) = \sum \delta(j) \omega^j$$

and

$$g(\omega^{2r}) = \sum \delta(j) \omega^{2rj} \quad \text{if } n \nmid 2r.$$

Hence, employing (3.13), we have

$$\frac{|f(\omega)|^h}{h!} - k^{h-1} \leq |g(\omega)| \leq \sum \delta(j)$$

and

$$\frac{|f(\omega^{2r})|^h}{h!} - k^{h-1} \leq |g(\omega^{2r})| \leq \sum \delta(j) \quad (n \nmid 2r).$$

Therefore

$$(3.16) \quad \sum \delta(j) + k^{h-1} \geq \frac{1}{h!} \max \{ |f(\omega)|^h, |f(\omega^2)|^h, |f(\omega^4)|^h, \dots, \\ |f(\omega^{2r})|^h, \dots; n \nmid 2r \},$$

the maximum existing since the set is actually finite ($\omega^{2n+j} = \omega^j$).

We obtain a lower estimate for this maximum by employing the
 Fourier series

$$(3.17) \quad \frac{\pi}{2} \sin \theta + 2 \sum_{r=1}^{\infty} \frac{\cos 2r\theta}{4r^2 - 1} = \begin{cases} 1 & (0 \leq \theta \leq \pi), \\ 1 + \pi \sin \theta & (\pi \leq \theta \leq 2\pi). \end{cases}$$

We consider the sum

$$\begin{aligned}
 F &= \frac{\pi}{2} \operatorname{Im} f(\omega) + 2 \sum_{\substack{r=1 \\ n \nmid 2r}}^{\infty} \frac{\operatorname{Re} f(\omega^{2r})}{4r^2 - 1} \\
 &= \sum_{j=1}^k \left(\frac{\pi}{2} \sin \frac{2\pi a_j}{n} + 2 \sum_{\substack{r=1 \\ n \nmid 2r}}^{\infty} \frac{\cos(4\pi r a_j / n)}{4r^2 - 1} \right) \\
 &= \sum_{j=1}^k \left(\frac{\pi}{2} \sin \frac{2\pi a_j}{n} + 2 \sum_{r=1}^{\infty} \frac{\cos(4\pi r a_j / n)}{4r^2 - 1} \right) - 2 \sum_{j=1}^k \sum_{\substack{r=1 \\ n \mid 2r}}^{\infty} \frac{1}{4r^2 - 1}.
 \end{aligned}$$

Now let λ_k denote the number of basis-elements in the interval $(n/2, n]$. Then by (3.17),

$$\begin{aligned}
 F &> k(1-\lambda) + (1-\pi)\lambda k - 2k \sum_{\substack{r=1 \\ n \nmid 2r}}^{\infty} \frac{1}{(2r)^2 - 1} \\
 &\geq k(1 - \lambda\pi - 2 \sum_{m=1}^{\infty} 1/[(mn)^2 - 1]) \\
 &> k(1 - \pi\lambda - 2/(n^2-1) \sum_{m=1}^{\infty} 1/m^2) \\
 &= k(1 - \pi\lambda - \pi^2/[3(n^2-1)]).
 \end{aligned}$$

Let $M = \max\{|f(\omega)|, |f(\omega^2)|, |f(\omega^4)|, \dots, |f(\omega^{2r})|, \dots; n \nmid 2r\}$.

Since the sum defining F is increased if we replace $\operatorname{Im} f(\omega)$ and $\operatorname{Re} f(\omega^{2r})$ ($r = 1, 2, \dots; n \nmid 2r$) by M , we have

$$F \leq M[\pi/2 + 2 \sum_{\substack{r=1 \\ n \nmid 2r}}^{\infty} 1/(4r^2 - 1)]$$

$$< M(\pi/2 + 2 \sum_{r=1}^{\infty} 1/(4r^2 - 1)) = M(1 + \pi/2).$$

Hence

$$M > k \frac{1 - \pi\lambda - \pi^2/[3(n^2-1)]}{1 + \pi/2}$$

and therefore from (3.16)

$$\sum \delta(j) + k^{h-1} \geq \frac{M^h}{h!} > \frac{k^h}{h!} \left(\frac{1 - \pi\lambda - \pi^2/[3(n^2-1)]}{1 + \pi/2} \right)^h.$$

Therefore by (3.15)

$$\frac{k^h}{h!} + 2k^{h-1} > n + \frac{k^h}{h!} \left(\frac{1 - \pi\lambda - \pi^2/[3(n^2-1)]}{1 + \pi/2} \right)^h$$

or

$$\frac{k^h}{h!} \left\{ 1 + \frac{2h!}{k} - \left(\frac{1 - \pi\lambda - \pi^2/[3(n^2-1)]}{1 + \pi/2} \right)^h \right\} > n.$$

Since we know that $1/k = O(1/n^{1/h})$, from (3.3), for example, we can assert the following lemma:

Lemma 3.1 If $\epsilon > 0$ is given, then

$$(3.18) \quad \frac{k^h}{h!} \left\{ 1 + \epsilon - \left(\frac{1 - \pi\lambda}{1 + \pi/2} \right)^h \right\} > n$$

provided n is sufficiently large.

If λ is small we shall use Lemma 3.1. On the other hand, let αk denote the number of basis-elements in the interval $(n/h, n]$. Then of the $\binom{k+h-1}{h}$ sums that can be formed using h summands from A , at least

$$\binom{\alpha k + h - 1}{h} + k(1-\alpha) \binom{\lambda k + h - 2}{h-1} + \binom{k(1-\alpha) + 1}{2} \binom{\lambda k + h - 3}{h-2} + \dots + \binom{k(1-\alpha) + h - 3}{h-2} \binom{\lambda k + 1}{2}$$

sums exceed n . This sum exceeds

$$\begin{aligned} & \frac{(\alpha k)^h}{h!} + k(1-\alpha) \frac{(\lambda k)^{h-1}}{(h-1)!} + \frac{k^2(1-\alpha)^2}{2!} \frac{(\lambda k)^{h-2}}{(h-2)!} + \dots + \frac{k^{h-2}(1-\alpha)^{h-2}}{(h-2)!} \frac{(\lambda k)^2}{2!} \\ &= \frac{k^h}{h!} \left(\alpha^h + h(1-\alpha)\lambda^{h-1} + \binom{h}{2}(1-\alpha)^2 \lambda^{h-2} + \dots + \binom{h}{h-2}(1-\alpha)^{h-2} \lambda^2 \right). \end{aligned}$$

Let us denote the expression in parentheses here by $W_h(\alpha, \lambda)$. Then

$$\binom{k+h-1}{h} > n + \frac{k^h}{h!} W_h(\alpha, \lambda),$$

and since
$$\binom{k+h-1}{h} < \frac{(k+h-1)^h}{h!} = \frac{k^h}{h!} \left(1 + \frac{h-1}{k} \right)^h,$$

$$\frac{k^h}{h!} \left\{ \left(1 + \frac{h-1}{k} \right)^h - W_h(\alpha, \lambda) \right\} > n.$$

Hence we can assert

Lemma 3.2 If $\epsilon > 0$ is given, then

$$(3.19) \quad \frac{k^h}{h!} \{1 + \epsilon - W_h(\alpha, \lambda)\} > n$$

provided n is sufficiently large.

Proof of (i) In the case $h = 2$, we have $\alpha = \lambda$ and $W_2(\alpha, \lambda) = \alpha^2$. The optimum result is obtained for

$$\alpha = \frac{1 - \pi\alpha}{1 + \pi/2} \quad \text{or} \quad \alpha = \alpha_0 = \frac{1}{1 + 3\pi/2} = .175058 \dots$$

If $\alpha < \alpha_0$, Lemma 3.1 yields the result that for any $\epsilon > 0$,

$$\frac{k^2}{2} (1 + \epsilon - .03064) > n$$

if n is sufficiently large, while if $\alpha \geq \alpha_0$, this follows from Lemma 3.2. Theorem 3.1 (i) follows.

Before proceeding to the proof of part (ii), we remark that in general, the optimal values of α and λ satisfy

$$W_h(\alpha, \lambda) = \left(\frac{1 - \pi\lambda}{1 + \pi/2} \right)^h$$

where the common value of these quantities is as large as possible. In the cases $h = 3$ and 4 , we shall find approximate optimal solutions of this equation by numerical trials. The method seems impracticable for high values of h , and for $h \geq 5$ we content ourselves with proving (3.11) and (3.12).

Proof of (ii) We write $W = W_3(\alpha, \lambda) = \alpha^3 + 3(1-\alpha)\lambda^2$. Then

$\frac{\partial W}{\partial \alpha} = 3\alpha^2 - 3\lambda^2$ and $\frac{\partial^2 W}{\partial \alpha^2} = 6\alpha \geq 0$, so that for a given value of λ , W is minimal when $\alpha = \lambda$. (Recall that $\alpha \geq \lambda$ by definition.) Hence

$$W \geq \lambda^3 + 3(1-\lambda)\lambda^2 = \lambda^2(3 - 2\lambda).$$

Note that this increases with λ , while $(1-\pi\lambda)/(1+\pi/2)$ increases as λ decreases. We find that for $\lambda = .0886$,

$$\left(\frac{1 - \pi\lambda}{1 + \pi/2} \right)^3 = .02212 \dots \quad \text{and} \quad \lambda^2(3 - 2\lambda) = .02215 \dots$$

Hence if $\lambda < .0886$, Lemma 3.1 implies (3.9) while if $\lambda \geq .0886$, Lemma 3.2 implies (3.9).

Proof of (iii) We write

$$W = W_4(\alpha, \lambda) = \alpha^4 + 4(1-\alpha)\lambda^3 + 6(1-\alpha)^2\lambda^2.$$

Again we use numerical trials to find a nearly optimal result.

For $\lambda = .05$,

$$(3.20) \quad \left(\frac{1 - \pi\lambda}{1 + \pi/2} \right)^4 > .01155,$$

and this inequality continues to hold for $\lambda < .05$.

Now let $\lambda \geq .05$. Then

$$W \geq \alpha^4 + .0005(1-\alpha) + .015(1-\alpha)^2.$$

Denoting this expression by $w(\alpha)$, we find

$$w'(\alpha) = 4\alpha^3 + .03\alpha - .0305$$

and

$$w''(\alpha) = 12\alpha^2 + .03.$$

Since $w''(\alpha) > 0$ for any α , w has an absolute minimum at some value α_0 of α . By numerical trial we find that this zero α_0 of $w'(\alpha)$ satisfies

$$.184 < \alpha_0 < .185.$$

Now we find

$$w(.183) \doteq .0115423,$$

$$w(.184) \doteq .0115420,$$

these estimates being correct to the number of decimal places shown. Because of the convexity of w , it is now obvious that

$$w(\alpha_0) > .0115415.$$

Hence, if $\lambda \geq .05$,

$$(3.21) \quad W > .011541.$$

(3.10) now follows by Lemmas 3.1 and 3.2.

Regarding the optimality of this result, we remark that we cannot increase the fourth decimal place in (3.20) and (3.21). We could increase the fourth decimal place in (3.20) only by using some $\lambda_1 < .05$. But since for given α , $W_4(\alpha, \lambda)$ increases with λ , we would find that

$$W_4(\alpha_0, \lambda_1) < W_4(\alpha_0, .05),$$

and since

$$\min_{\alpha} W_4(\alpha, \lambda_1) < W_4(\alpha_0, \lambda_1),$$

we would not be able to assert, in place of (3.21), that $W > .0116$ if $\lambda \geq \lambda_1$. Hence we cannot improve the fourth decimal place of the number .01154 shown in (3.10) by the method of this chapter.

Proof of (iv) and (v) Let us abbreviate $s(h)$ to s .

For any $\epsilon > 0$, provided n is sufficiently large, Lemma 3.1 will yield

$$(3.22) \quad \frac{k^h}{h!} \{1 + \epsilon - [(1+\beta)s]^h\} > n$$

if $\frac{1 - \pi\lambda}{1 + \pi/2} > (1+\beta)s$, or if $\pi\lambda < 1 - (1+\pi/2)(1+\beta)s$.

(For the moment we regard β as some suitable non-negative number.)

Assume on the other hand that

$$(3.23) \quad \pi\lambda \geq 1 - (1+\pi/2)(1+\beta)s.$$

We wish to show that

$$(3.24) \quad W_h(\alpha, \lambda) > [(1+\beta)s]^h$$

for some value of β , so that Lemma 3.2 will yield (3.22).

(3.23) implies $\pi\lambda > [3 - (1+\pi/2)(1+\beta)]s$ since $s < 1/3$

for any h . We write $c = 3 - (1+\pi/2)(1+\beta)$, so that

$$(3.25) \quad \lambda > c s / \pi.$$

Then

$$W_h(\alpha, \lambda) > \alpha^{h+h(1-\alpha)} \left(\frac{cs}{\pi} \right)^{h-1} + \binom{h}{2} (1-\alpha)^2 \left(\frac{cs}{\pi} \right)^{h-2} + \dots + \binom{h}{h-2} (1-\alpha)^{h-2} \left(\frac{cs}{\pi} \right)^2$$

and this exceeds $[(1+\beta)s]^h$ if and only if

$$(3.26) \quad \left(\frac{\alpha}{(1+\beta)s} \right)^h + h \frac{1-\alpha}{(1+\beta)s} \left(\frac{c}{(1+\beta)\pi} \right)^{h-1} + \binom{h}{2} \left(\frac{1-\alpha}{(1+\beta)s} \right)^2 \left(\frac{c}{(1+\beta)\pi} \right)^{h-2} + \dots \\ + \binom{h}{h-2} \left(\frac{1-\alpha}{(1+\beta)s} \right)^{h-2} \left(\frac{c}{(1+\beta)\pi} \right)^2 > 1.$$

Since $\alpha \geq \lambda$, $\alpha/s \geq \lambda/s > c/\pi$. Furthermore, if $\alpha \geq (1+\beta)s$, then (3.24) would follow, so that for the purpose of proving (3.26) we can assume $\alpha < (1+\beta)s$.

Then

$$(1-\alpha)/s > 1/s - (1+\beta) > 3 - (1+\beta) = 2 - \beta.$$

Hence the left side of (3.26) exceeds

$$(3.27) \quad \left(\frac{c}{(1+\beta)\pi} \right)^h + h \frac{2-\beta}{1+\beta} \left(\frac{c}{(1+\beta)\pi} \right)^{h-1} + \dots + \binom{h}{h-2} \left(\frac{2-\beta}{1+\beta} \right)^{h-2} \left(\frac{c}{(1+\beta)\pi} \right)^2.$$

A straightforward calculation shows that for $\beta = .0201$ and $h \geq 5$, the last term alone of (3.27) exceeds 1, while for $\beta = .101$ and $h \geq 8$ the last term exceeds 1. Hence for these respective values of β and h , (3.24) holds, and Lemma 3.2 yields (3.22).

Thus (3.22) holds in any case for these respective values of β and h , and from this we obtain respectively (3.11) and (3.12)

The proof of theorem 3.1 is now complete.

We would like to remark that using the whole of the sum (3.27) would allow but little improvement. The sum can be written

$$\left(\frac{2-\beta}{1+\beta}\right)^h \left[\left(1 + \frac{1}{\pi} - \frac{1}{2} \frac{1+\beta}{2-\beta}\right)^h - h \left(\frac{1}{\pi} - \frac{1}{2} \frac{1+\beta}{2-\beta}\right) - 1 \right]$$

and we find that for $\beta = .03$ and $h = 5$, this is less than 1, and for $\beta = .1$, $h = 7$, it is again less than 1. The last term of (3.27) would seem to be the bulk of the sum.

§3.3 The restricted case

Here, all k of the basis-elements lie in the interval $[0, n/h]$. We prove the following theorems:

Theorem 3.2 Let $h = 2$. Then for n sufficiently large,

$$(3.28) \quad \frac{k^2}{2} (1 - .1513) > n.$$

Theorem 3.3 Let $h \geq 3$. For any $\epsilon > 0$,

$$(3.29) \quad \frac{k^h}{h!} > \frac{2^{h-1}}{h(1+\epsilon)} n$$

provided n is sufficiently large.

Proof of Theorem 3.2 When $h = 2$, $\alpha = \lambda = 0$ and the theorem follows directly from Lemma 3.1.

Lemma 3.1 would still yield a result for $h \geq 3$, but it would not be as good as the previous lower estimate (3.7). This is easy to see since for $h \geq 3$,

$$\cos(\pi/h) \geq 1/2 > 1/(1+\pi/2).$$

Proof of Theorem 3.3 Let there be m_1 basis-elements in $[0, n/2h]$ and m_2 basis-elements in $(n/2h, n/h]$. Since the integers in $[0, n/2h]$ can be represented only as sums of basis-elements from that interval, it follows that

$$(3.30) \quad \binom{m_1 + h - 1}{h} \geq \left\lfloor \frac{n}{2h} \right\rfloor + 1 > \frac{n}{2h}.$$

Further, the largest integer that can be represented if one of the h summands is in $[0, n/2h]$ does not exceed $(h-1)n/h + n/2h = n - n/2h$. Hence the integers in $(n - n/2h, n]$ can be represented only as sums of basis elements from $(n/2h, n/h]$, and therefore

$$(3.31) \quad \binom{m_2 + h - 1}{h} \geq n - [n - n/2h] \geq n - (n - n/2h) = n/2h.$$

Hence

$$\frac{(m_i + h - 1)^h}{h!} > \frac{n}{2h} \quad \text{for } i = 1, 2.$$

Since either $m_1 \leq k/2$ or $m_2 \leq k/2$, it follows that

$$\frac{(k/2)^h}{h!} \left(1 + \frac{2(h-1)}{k} \right)^h > \frac{n}{2h},$$

and Theorem 3.3 follows.

Comparison of (3.29) with (3.7) We write (3.7) in the form

$$\frac{k^h}{h!} > \frac{n}{1 + \varepsilon - [\cos(\pi/h)]^h}$$

and compare the right side here with that in (3.29). The inequality

$$\frac{2^{h-1}}{h(1+\varepsilon)} > \frac{1}{1 + \varepsilon - [\cos(\pi/h)]^h}$$

is equivalent to

$$\frac{2^{h-1}}{h} > \frac{1}{1 - [\cos(\pi/h)]^h / (1+\varepsilon)},$$

and this is true since, as we shall prove for $h \geq 3$,

$$(3.32) \quad \frac{2^{h-1}}{h} > \frac{1}{1 - [\cos(\pi/h)]^h}.$$

The comparison for $h = 3$ and $h = 4$ is given in the following table:

h	$2^{h-1}/h$	$\frac{1}{1 - [\cos(\pi/h)]^h}$
3	4/3	8/7
4	2	4/3

For $h \geq 5$ we not only prove (3.32), but show

$$(3.33) \quad \frac{1}{1 - [\cos(\pi/h)]^h} < \frac{h}{3}.$$

This inequality seems interesting since $2^{h-1}/h$ exceeds $h/3$ by a very substantial amount; the difference in order of magnitude of the quantities in (3.32) is correctly illustrated by this comparison since

$$\frac{1}{1 - [\cos(\pi/h)]^h} \sim \frac{2h}{\pi^2} \doteq \frac{h}{4.9}.$$

The inequality (3.33) is equivalent to

$$[\cos(\pi/h)]^h < 1 - 3/h.$$

Now,

$$\cos(\pi/h) < 1 - \frac{\pi^2}{2h^2} \left(1 - \frac{\pi^2}{12h^2} \right) < 1 - \frac{\pi^2}{2h^2} \cdot \frac{29}{30} < 1 - \frac{87}{20h^2}$$

since for $h \geq 5$, $1 - \pi^2/12h^2 > 1 - 10/(12 \cdot 25) = 29/30$.

Therefore we now wish to prove

$$(3.34) \quad \left(1 - \frac{4.35}{h^2} \right)^h < 1 - \frac{3}{h},$$

and (3.33) will follow.

For $h = 5$, the left side of (3.34) is $(.826)^5 < .39 < 1-3/5$, and we turn to the case $h \geq 6$. First we observe that

$$\left(1 - \frac{4.35}{h^2}\right)^h < 1 - h \frac{4.35}{h^2} + \binom{h}{2} \left(\frac{4.35}{h^2}\right)^2$$

since the terms in the binomial expansion decrease in absolute value. Next,

$$1 - \frac{4.35}{h} + \frac{h-1}{2} \frac{(4.35)^2}{h^3} < 1 - \frac{3}{h}$$

is equivalent to

$$9.46125 \frac{h-1}{h^2} < 1.35$$

and one easily shows that this holds for $h \geq 6$. Thence (3.33).

§3.4 On bases for $Z = \{0, 1, 2, \dots\}$

In the following we let B denote an h -basis for Z . $B(n)$, the counting function of B , denotes the number of non-zero elements of B not exceeding n .

Stöhr [20] considered the problem of estimating $\liminf_{n \rightarrow \infty} \frac{B(n)}{n^{1/h}}$, and by a simple combinatorial argument (essentially like the derivation of (3.3) from (3.2)) showed that for any h -basis B ,

$$(3.35) \quad \liminf_{n \rightarrow \infty} \frac{B(n)}{n^{1/h}} \geq (h!)^{1/h}.$$

He used the result (3.4 a) of Rohrbach to obtain in the case $h = 2$ the sharper estimate

$$(3.36) \quad \liminf_{n \rightarrow \infty} \frac{B(n)}{\sqrt{n}} \geq \frac{1}{\sqrt{.4992}}.$$

This obtains since the elements not exceeding n in any 2-basis B must form a 2-basis for n , so that

$$\frac{B(n)}{\sqrt{n}} \geq \frac{k_2(n) - 1}{\sqrt{n}}. \quad \text{Similarly, for any } h\text{-basis } B,$$

$$\frac{B(n)}{n^{1/h}} \geq \frac{k_h(n) - 1}{n^{1/h}}.$$

(The "-1" appears since $B(n)$ counts non-zero elements and $k_h(n)$ counts 0 as well.) At the time that Stöhr wrote his paper, (3.35) was the best lower estimate to be had in case $h \geq 3$. Our estimates (3.8) through (3.12) in Theorem 3.1 allow us to improve on (3.35) and (3.36). It will be convenient to consider (3.8) through (3.12) in the form

$$\frac{k_h^h(n)}{h!} (1 - \beta_h) > n,$$

with

$$\beta_2 = .0306, \quad \beta_3 = .0221, \quad \beta_4 = .01154,$$

$$(3.37) \quad \beta_h = (1.02 s(h))^h \text{ for } h = 5, 6, 7, \text{ and}$$

$$\beta_h = (1.1 s(h))^h \text{ for } h \geq 8.$$

(Recall that $s(h) = \cos(\pi/h)/(2 + \cos(\pi/h))$.)

Then we can assert for $h \geq 2$ and any h -basis B ,

$$(3.38) \quad \liminf_{n \rightarrow \infty} \frac{B(n)}{n^{1/h}} \geq \left(\frac{h!}{1 - \beta_h} \right)^{1/h}$$

For $h = 2$ this estimate is $1/\sqrt{.4847}$.

Theorem 3.1 also allows a more precise statement of a theorem of E. Härtter [8]. Let $b_0 = 0 < b_1 < b_2 < \dots$ denote the elements of an h -basis B . Härtter used the result of Stöhr that for any h -basis B ,

$$\limsup_{n \rightarrow \infty} \frac{B(n)}{n^{1/h}} \leq \frac{(h!)^{1/h}}{\Gamma(1 + 1/h)}$$

[20, inequality (5)], to prove the following:

There is no h -basis B ($h \geq 2$) such that

$$(3.39) \quad b_k \geq \frac{k^h}{h!} + t(k) \quad \text{for all } k > K$$

for any sequence $\{t(k)\}$, $0 \leq t(k) = o(k^h)$.

Since in B the elements $b_0, b_1, \dots, b_k$ must form an h -basis for b_k , it follows from Theorem 3.1 that

$$(3.40) \quad b_k < \frac{(k+1)^h}{h!} (1 - \beta_h)$$

for all sufficiently large k .

We could write k in place of the $k + 1$ on the right side here by making a small adjustment in the proof of Theorem 3.1, where we found it convenient to let k be the total number of basis-elements, including 0 . Clearly (3.40) says more than (3.39) in that (3.40) puts a positive lower bound on $k^h/h! - b_k$ when k is large.

Härtter's paper also contains a theorem pertaining to 2-bases. He defines $\Delta_k = b_{k+1} - b_k$ ($k = 0, 1, \dots$) and proves that there is no 2-basis such that $\Delta_{k+1} > \Delta_k$ for all large k . We make this more precise.

Suppose that $\Delta_{k+1} > c \Delta_k$ for all $k \geq K$ where $c > 0$. Härtter shows that this is impossible with $c \geq 1$. Therefore $0 < c < 1$ and we shall show that for any $\epsilon > 0$, there is some K_0 such that

$$(3.41) \quad c < 1 - \frac{\Delta_K (1 - \epsilon)}{.4847 K_0^2 - b_K}.$$

For,

$$\begin{aligned} b_k &= b_K + \Delta_K + \Delta_{K+1} + \dots + \Delta_{k-1} \\ &> b_K + \Delta_K + c\Delta_K + c^2\Delta_K + \dots + c^{k-K-1}\Delta_K \\ &= b_K + \Delta_K \frac{1 - c^{k-K}}{1 - c} \\ &> b_K + \frac{1 - \epsilon}{1 - c} \Delta_K \end{aligned}$$

if k is sufficiently large. Hence, from (3.8), there is

some K_0 such that

$$(3.42) \quad .4847 (k+1)^2 > b_K + \frac{1-\varepsilon}{1-c} \Delta_K$$

for all $k \geq K_0 - 1$. In particular we have (3.42) with $k = K_0 - 1$, and hence (3.41).

Lastly we discuss briefly an implication of Theorem 3.1 as it relates to some results of Cassels [4]. Cassels proved:

For each $h \geq 2$ there is an h -basis B such that

$$(3.43) \quad \lim_{k \rightarrow \infty} \frac{b_k}{k^h} \text{ exists and is positive.}$$

Furthermore, among such h -bases there is one for which

$$b_k = \alpha_h k^h + O(k^{h-1}).$$

For $h = 2$ there exist bases B and C such that

$$\liminf_{k \rightarrow \infty} \frac{b_k}{k^2} = \frac{8}{71 + 17^{3/2}} \doteq \frac{3}{53}$$

and

$$\lim_{k \rightarrow \infty} \frac{c_k}{k^2} = \frac{1}{27}.$$

For $h > 2$ he found no specific value for the limit in (3.43).

Cassels also proved:

$$\text{For every 2-basis } B, \quad \liminf_{k \rightarrow \infty} \frac{b_k}{k^2} \leq \frac{\pi}{8} < .3927.$$

Cassels mentioned no estimate for the upper limit of b_k/k^h , which at that time could only be given as not exceeding $1/h!$, with a slight reduction in the case $h = 2$. We shall state an upper limit that is implied by Theorem 3.1.

If B is an h -basis and β_h is the quantity defined by (3.37), then from Theorem 3.1,

$$\frac{b_k}{(k+1)^h} < \frac{1 - \beta_h}{h!}$$

for all sufficiently large k . Hence we can assert that for any h -basis B ,

$$\limsup_{k \rightarrow \infty} \frac{b_k}{k^h} \leq \frac{1 - \beta_h}{h!}.$$

This inequality is equivalent to (3.38), since

$$\liminf_{n \rightarrow \infty} \frac{B(n)}{n^{1/h}} = \liminf_{k \rightarrow \infty} \frac{k}{b_k^{1/h}} \geq \gamma$$

is equivalent to

$$\limsup_{k \rightarrow \infty} \frac{b_k}{k^h} \leq \frac{1}{\gamma^h}.$$

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APPENDIX

ESTIMATION OF $D(Q(E_4'))$

From § 2.2,

$$D(Q(E_4')) > 1 - \sum_p K(p) + \sum_{p < q} K(p) K(q) - \sum_{p < q < r} K(p) K(q) K(r).$$

Upper estimate for $\sum_p K(p)$

p	$K(p) = (p^7 - p^6 + p^4 - p^3 + 1)/p^{10}$		
2	73/1,024	÷	.07128 90625
3	1,513/59,049		.02562 27878
5	63,001/9,765,625		.00645 13024
7	707,953/282,475,249		.00250 62479
11	17,728,921/25,937,424,601		.00068 35267
13	57,948,073/(137.858487 x 10 ⁹)		.00042 03446
17	386,279,713/(2,015.99379 x 10 ⁹)		.00019 16075
19	846,949,321/(6,131.06611 x 10 ⁹)		.00013 81406
23	3,257,056,786/(414.265121 x 10 ¹¹)		.00007 86225
29	16,655,735,572/(4,207.07234 x 10 ¹¹)		.00003 95899
			.10742 12324

Hence $\sum_{p < 29} K(p) \div .10742 \ 12324$ correct to at least six decimal places. Now, since $K(p) < 1/p^3$,

$$\sum_{p > 29} K(p) < \sum_{31 \leq p \leq 97} 1/p^3 + \sum_{p \geq 101} 1/p^3$$

$$(1) \qquad \qquad \qquad < .000122 \ + \ .000025 \ = \ .000147$$

where the first sum is estimated from tables and the second by an integral:

$$\sum_{p \geq 101} \frac{1}{p^3} < \sum_{m=50}^{\infty} \frac{1}{(2m+1)^3} < \int_{49.5}^{\infty} \frac{dx}{(2x+1)^3} = .000025.$$

Hence

$$(2) \quad \sum_p K(p) < .107422 + .000147 = .107569.$$

Lower estimate for $\sum_{p < q} K(p) K(q)$

From the previous table we obtain the following one:

<u>p</u>	<u>K(2) K(p)</u>	<u>K(3) K(p)</u>	<u>K(5) K(p)</u>	<u>K(7) K(p)</u>
3	.00182 662	--	--	--
5	.00045 990	.00016 530	--	--
7	.00017 866	.00006 421	.00001 616	--
11	.00004 872	.00001 751	.00000 440	.00000 171
13	.00002 996	.00001 077	.00000 271	.00000 105
17	.00001 365	.00000 490	.00000 123	.00000 048
19	.00000 984	.00000 353	.00000 089	.00000 034
23	.00000 560	.00000 201	.00000 050	.00000 019
29	.00000 282	.00000 101	.00000 025	.00000 009
Totals:	.00257 577	.00026 924	.00002 614	.00000 386

Hence

$$(3) \quad \sum_{p < q} K(p) K(q) > .00287 501 > .002875$$

Upper estimate for $\sum_{p < q < r} K(p) K(q) K(r)$

From the previous tables and (1) we obtain the following table:

<u>p</u>	<u>K(2) K(3) K(p) <</u>	<u>K(2) K(5) K(p) <</u>	<u>K(3) K(5) K(p) <</u>
5	.00001 179	--	--
7	.00000 458	.00000 116	.00000 042
11	.00000 125	$\sum_{p \geq 11} K(2) K(5) K(p) <$	$\sum_{p \geq 11} K(3) K(5) K(p) <$
13	.00000 077	.00000 079	.00000 029
17	.00000 036		
19	.00000 026		
	$\sum_{p \geq 23} K(2) K(3) K(p) <$		
	<u>.00000 049</u>		
Totals:	.00001 950	.00000 195	.00000 071

Hence

$$\sum_{p < q < r} K(p) K(q) K(r) < .00002216 + \{K(2) + K(3)\} \sum_{7 \leq p < q} K(p) K(q)$$

$$+ \sum_{5 \leq p < q < r} K(p) K(q) K(r)$$

$$< .00002216 + \{K(2) + K(3)\} \left(\sum_{p \geq 7} K(p) \right) \left(\sum_{q \geq 11} K(q) \right) \\ + \left(\sum_{p \geq 5} K(p) \right) \left(\sum_{q \geq 7} K(q) \right) \left(\sum_{r \geq 11} K(r) \right)$$

$$< .00002216 + \left(\sum_p K(p) \right) \left(\sum_{q \geq 7} K(q) \right) \left(\sum_{r \geq 11} K(r) \right)$$

$$< .00002216 + (.10742123) (.00420508) (.00169884)$$

$$(4) \quad < .00002216 + .00000077 = .00002293 < .000023.$$

Hence from (2), (3), and (4),

$$D(Q(E_4')) > 1 - .107569 + .002875 - .000023 > .8952.$$

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